Restriction of Preferences to the Set of Consumption Bundles, In a Model with Production and Consumption Bundles

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Abstract

In contrast to the neo-classical theory of Arrow and Debreu, a model of a private ownership economy is presented, in which production and consumption bundles are treated separately. Each of the two types of bundles is assumed to establish a convex cone. Production technologies can convert production bundles into consumption bundles, and the preferences of the consumers are assumed to lie only on the set of consumption bundles. The main theorem of this paper states the existence of a Walrasian equilibrium in this setting.

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Introduction

In [9], a model of a pure exchange economy is presented only in terms of convex cones, whereas, in case of n commodities, the neo-classical models are set in terms of the Euclidean space $\mathbb{R}^n$. As a consequence, the model of [9] recognises economy bundles, rather than separate commodities, as entities of exchange. The model of a private ownership economy, presented in [11], introduces production in this setting. Apart from the use of convex cones, the model of [11] differs from the neo-classical models described in the standard works [3] and [1], since in this model distinction is made between production and consumption bundles.

The use of convex cones is emphasized by an axiomatic introduction of the concept of salient half-space, a set in which addition and scalar multiplication over the positive reals are defined such that the set is an addition semi-group. The main difference with a vector space is that, for a salient half-space multiplication is allowed over the non-negative real numbers only, and that addition is not a group. Each pointed convex cone in a vector space is a salient half-space. Also, each salient half-space induces a partially ordered vector space for which the salient half-space is the positive cone. In [11], a great deal of effort is put in the presentation of this mathematical concept of salient half-space and related topics. Also in [11], these concepts are incorporated in a model of a private ownership economy.

In the first section of this paper we further investigate the concept of salient half-space. Among the new contributions are the definitions of subcone, convex cone span, finitely generated salient half-space, cone-dependence and finite-dimensional salient half-space. The existence of a maximal cone-independent set is proved and its relation with the vector space reproduced by the salient half-space is given. Furthermore, the concept of interior point is introduced, a relation between interior points and order units is given, and there is an introduction of a metric, corresponding to an order unit of the half-dual space, which induces a topology on the salient half-space.

Section 2 of this paper is devoted to the introduction of a model of a private ownership economy and a suitable definition of a Walrasian equilibrium in this setting. The model presented here is an adaption of the model described in [11].

Following [11], we do not introduce the concept of "commodity" but consider the concept of "economy bundle", which carries the characteristics of exchangable objects in the economy, instead. An economy bundle is a unique concatenation of a production (economy) bundle and a consumption (economy) bundle. Only production bundles can be used as input for a production process whereas the output of this process is always a consumption bundle. The set consisting of all economy bundles is taken to be the direct sum of the salient half-spaces C_{prod} and C_{cons}, containing the production and consumption bundles, respectively. This direct sum $C_{\text{prod}} \oplus C_{\text{cons}}$ is a salient half-space, and will often be denoted by C.

The main difference between the model of a private ownership economy described in [11] and the model described in Section 2 of this paper, is the assumption that the preference relations of the consumers are only defined on the set C_{cons} of consumption (economy) bundles, instead of on the direct sum $C_{\text{prod}} \oplus C_{\text{cons}}$. This reflects a disinterest of consumers for production bundles, following from their inabillity to consume production bundles and from the absence of a “second time period” in which the consumers are able to sell purchased production bundles.

In a worldlike example, our model can describe the non-neo-classical situation in which fixed links between different commodities are present, for instance an economy in which only fixed, prescribed combinations of commodities can be traded. Examples are special pre-packed offers, or free (sample)-products received when purchasing a commodity. Also, this model can describe a situation in which the preferences of the agents are in terms of characteristics of commodities, instead of in terms of the commodities themselves. In the labour market, for instance, a firm may ask for an employee with a certain education, intelligence and working experience. In this setting, one can consider an “economy bundle” to be a person with such (and perhaps other) specific attributes. In general, an “economy bundle” can be considered to be a carrier of several attributes (cf. the work of Lancaster, [6]). Moreover, the same attribute may appear in more than one economy bundle. This mixture of attributes can be inextricable both in characteristics and in time.

In Section 3, the concept of production technology, introduced in Section 2, is explored and, in comparison with [10], some new properties are derived regarding the closedness of a production technology and the closedness of its efficient set. Furthermore, some assumptions are introduced, which imply that we can deal with supply functions instead of with supply sets. Also, these assumptions imply that the supply functions are continuous on a domain which possibly is larger than the domain defined in [10]. More specifically, this new approach allows for zero prices for certain production (economy) bundles. Typically, production bundles which can only be used to produce certain consumption bundles, for which there is a cheaper way of producing them, will have zero-prices. As a consequence, several proofs of the stated lemmas and propositions differ from the corresponding ones in [10].

Section 3 ends with the definiton of the total supply function and with the derivation of some lemmas concerning production technologies and supply functions which will be useful in Section 5, where the existence of a Walrasian equilibrium pricing function is proved.

The arrangement of Section 4 is similar to the one of Section 3, only here the topic concerns the agents instead of the production technologies. Firstly, the concepts of budget set and demand set, defined in Section 2, are further explored. Some assumptions which guarantee continuous demand functions on (a subset of) the domain of the total supply function, are introduced. The total demand function is defined and, finally, some useful lemmas will be derived. The main difference with the corresponding section of [10] will be the restriction of most concepts to the salient half-space C_{cons} of consumption (econcomy) bundles. This is a consequence of the restriction of the preference relations of the agents to C_{cons}

The Equilibrium Existence Theorem, the main theorem of this paper, is presented in Section 5. It states that under rather weak conditions, among which the assumptions introduced in Sections 2 and 3, the existence of a Walrasian equilibrium pricing function is guaranteed. The paper ends with a proof of this main theorem.

1 Mathematical concepts

One of the essential differences between the model of a private ownership economy in this paper and the well-known models of Arrow and Debreu (cf. [1] and [3]) is the use of so-called salient half-spaces to describe the set of all exchangable objects and the set of all prices. Before we start with the description of our model, which will be done in Section 2, we give a formal introduction to the concept of salient half-space and related topics. This section is divided into three parts. The first part introduces the concept of salient half-space and describes the construction of the vector space reproduced by it and the partial order relation induced by it. Also, other concepts closely related to salient half-spaces, such as the concept of cone span, of cone dependancy, of finitely generated salient half-space and of order unit, are introduced. The second part of this section focusses on the half-dual space, i.e., the dual set of a salient half-space. Finally, in the third part titled "Topology", we present conditions on the half-dual space which guarantee existence of a metric on the salient half-space. Also, several properties for finite-dimensional, reflexive salient half-spaces are derived, among which an adaption of the well-known Brouwer Fixed Point Theorem.

Salient half-space

As mentioned above, we start with the formal introduction of the concept of salient half-space and the vector space reproduced by it.

Definition 1.1 A *salient half-space* is a set C with the following properties:

- An addition is defined on C, which is commutative, associative and satisfies

  1.1.a) there exists an element $v \in C$, called the *vertex* of C, such that $x + y = v \iff x = y = v$,

  1.1.b) for every $x \in C$ the mapping $\text{add}_x : C \to C$, defined by $\text{add}_x(y) := y + x$, is injective.

- To every pair $x \in C$ and $\alpha \geq 0$, there corresponds an element $\alpha x \in C$, called the (scalar) product of α and x. Scalar multiplication over $I\!R^+$ thus defined, is associative and satisfies the distributive laws. Furthermore, $1x = x$ holds for every $x \in C$.

Note that Condition 1.1.a implies that the mapping add_x is surjective if and only if $x = v$. It is not difficult to prove that the vertex of a salient half-space is unique and satisfies

- $\forall \alpha > 0 \quad : \alpha v = v,$
- $\forall x \in C \quad : x + v = x,$
- $\forall x \in C \quad : 0x = v.$

From the second property together with the first conditions of Definition 1.1, we conclude that $(C, +)$ is an addition semi-group with zero-element v. Since in a salient half-space, scalar multiplication is defined only over $I\!R^+$ and due to Condition 1.1.a, $(C, +)$ is not a group. However, we can extend $(C, +)$ to a group in a similar way as $I\!N \cup \{0\}$ extends to $Z\!\!Z$, by defining the equivalence relation $\sim$ on the product set $C \times C$ by:

$$(x_1, x_2) \sim (y_1, y_2) :\iff x_1 + y_2 = y_1 + x_2.$$

Let $V[C]$ be the collection of all equivalent classes $[(y_1, y_2)] := \{(z_1, z_2) \in C \times C \mid (z_1, z_2) \sim (y_1, y_2)\}$, so $V[C] := (C \times C)/_{\sim}$. Unambiguously, we can define the following addition and scalar multiplication over $I\!R$ on $V[C]$:

$$\begin{aligned}
[(y_1, y_2)] + [(z_1, z_2)] &:= [(y_1 + z_1, y_2 + z_2)] \\
\alpha[(y_1, y_2)] &:= \begin{cases} [(\alpha y_1, \alpha y_2)] & \text{if } \alpha \geq 0 \\ [((-\alpha)y_2, (-\alpha)y_1)] & \text{if } \alpha < 0. \end{cases}
\end{aligned}$$

With these definitions, $V[C]$ becomes a real vector space. We call $V[C]$ the vector space reproduced by the salient half-space C. Identifying each element $x \in C$ with the vector $[(x, 0)] \in V[C]$, we regard the salient half-space C to be a subset of $V[C]$. Note that C is total in $V[C]$, i.e., the linear span of C, denoted by $\text{span}_{V[C]}(C)$, equals the vector space $V[C]$. The vertex v of C coincides with the origin of $V[C]$, and henceforward we shall denote the vertex of a salient half-space by 0. Also, C is a convex cone in $V[C]$, which means that C is a convex set that is closed under addition and scalar multiplication over $I\!R^+$.

Definition 1.2 A subset S of a salient half-space C is a *subcone* of C, if S, endowed with the addition and scalar multiplication over $I\!R^+$ of C, is a salient half-space.

Proposition 1.3 *A subset S of a salient half-space C is a subcone of C if and only if $\forall x, y \in S \; \forall \alpha \in I\!R^+ : x + y \in S$ and $\alpha x \in S$.*

Definition 1.4 For any subset S of a salient half-space C the *convex cone span* of S, denoted by $\text{cone}(S)$, is the intersection of all subcones of C, that contain S. If there is a finite set S such that $\text{cone}(S) = C$ then C is a *finitely generated* salient half-space.

Note that by definition, $\text{cone}(\emptyset) = \{0\}$. Furthermore, for every $S \subset C : V[\text{cone}(S)] = \text{span}_{V[C]}(S)$, and so $\text{cone}(S) \subset \text{span}_{V[C]}(S)$ where $\text{cone}(S)$ is regarded as a subset of $V[C]$. As a result, if S is a finite set in C, then $V[\text{cone}(S)]$ is a finite-dimensional subspace of $V[C]$.

The proof of the following proposition is similar to the proof of Theorem 2.3 of [7], stating that the convex hull of a set S consists of all (finite) convex combinations of the elements of S. Henceforth, we call an element of $\text{cone}(S)$, a (finite) cone combination of S.

Proposition 1.5 *Let S be a subset of a salient half-space C, then for every $x \in \text{cone}(S)$, there is a finite set $F \subset S$ such that $x \in \text{cone}(F)$. Hence,*

$$\text{cone}(S) = \{x \in C \mid \exists n \in I\!N \; \exists x_1, \ldots, x_n \in S \; \exists \lambda_1, \ldots, \lambda_n \in I\!R^+ : x = \sum_{i=1}^{n} \lambda_i x_i\}.$$

Definition 1.6 On a salient half-space C the partial *order relation* $\leq_C$ is given by

$$\begin{aligned}
x \leq_C y \quad &:\Longleftrightarrow \quad \exists z \in C : x + z = y, \\
x <_C y \quad &:\Longleftrightarrow \quad \exists z \in C \setminus \{0\} : x + z = y.
\end{aligned}$$

Above, we mentioned that a salient half-space C can be identified with $\{[(y_1, y_2)] \subset V[C] \mid \exists x \in C : [(y_1, y_2)] = [(x, 0)]\}$. The partial order relation $\leq_C$, defined on C, can be extended to a partial order relation $\leq$ on $V[C]$ by defining for all $[(y_1, y_2)], [(z_1, z_2)] \in V[C]$:

$$[(y_1, y_2)] \leq [(0, 0)] \quad :\Longleftrightarrow \quad y_1 \leq_C y_2$$

and

$$[(y_1, y_2)] \leq [(z_1, z_2)] \quad :\Longleftrightarrow \quad [(y_1 + z_2, y_2 + z_1)] \leq [(0,0)].$$

Note that this partial order relation on $V[C]$ satisfies:

$$[(y_1, y_2)] \leq [(z_1, z_2)] \quad \Longleftrightarrow \quad \exists [(x_1, x_2)] \in C : [(y_1, y_2)] + [(x_1, x_2)] = [(z_1, z_2)],$$

where C is regarded as a subset of $V[C]$.

Definition 1.7 An element u of a salient half-space C is an *order unit* (for C) if

$$\forall x \in C \; \exists \lambda \geq 0 : x \leq_C \lambda u.$$

Lemma 1.8 *Let u be an order unit for C, and let $[(y_1, y_2)] \in V[C]$. Then*

$$\exists \lambda \geq 0 : -\lambda[(u, 0)] \leq [(y_1, y_2)] \leq \lambda[(u, 0)].$$

Proof

Since u is an order unit for C, we find $\begin{cases} \exists \lambda_1 \geq 0 : y_1 \leq_C \lambda_1 u \\ \exists \lambda_2 \geq 0 : y_2 \leq_C \lambda_2 u. \end{cases}$

Define $\lambda := \max\{\lambda_1, \lambda_2\}$, then $\begin{cases} y_1 \leq_C y_2 + \lambda u \\ y_2 \leq_C y_1 + \lambda u. \end{cases}$ □

We next introduce the definition of internal point, and lay the relation with order unit.

Definition 1.9 Let S be a subset of a salient half-space C. Then an element $s_0 \in S$ is an *internal point* of S if $\forall x, y \in C \; \exists \varepsilon > 0 \; \forall \tau \in (0, \varepsilon) : s_0 + \tau x \in \{\tau y\} + S$.

When we consider S as a subset of the vector space $V[C]$, then an internal point s_0 of S, as defined above, is an internal point of S, in accordance with the definition in [2, Chapter IV.1].

Corollary 1.10 *Let C be a salient half-space and let $s_0 \in C$. Then s_0 is internal point of C if and only if $\forall y \in C \; \exists \varepsilon > 0 : s_0 \in \{\varepsilon y\} + C$.*

Proof

Clearly, the statement $\forall y \in C \; \exists \varepsilon > 0 : s_0 \in \{\varepsilon y\} + C$ is implied by the definition of an internal point of a salient half-space C. Furthermore, $\forall y \in C \; \exists \varepsilon > 0 : s_0 \in \{\varepsilon y\} + C$ is equivalent with $\forall x, y \in C \; \exists \varepsilon > 0 : s_0 + \varepsilon x \in \{\varepsilon y + \varepsilon x\} + C \subset \{\varepsilon y\} + C$. Since the set $\{\varepsilon y\} + C$ is convex and contains s_0, we conclude $\forall x, y \in C \; \exists \varepsilon > 0 \; \forall \tau \in (0, \varepsilon) : s_0 + \tau x \in \{\varepsilon y\} + C \subseteq \{\tau y\} + C$. □

By definition, $u \in C$ is an order unit if and only if $\forall x \in C \; \exists \lambda > 0 : u \in \{\frac{1}{\lambda} x\} + C$. This proves the following lemma.

Lemma 1.11 *Let C be a salient half-space and let $u \in C$. Then u is an order unit in C if and only if u is an internal point of C.*

The set of all internal points of a salient half-space C will be denoted by $\mathrm{int}(C)$. Using the definition of order unit, it is not difficult to check that $\mathrm{int}(C) \cup \{0\}$ is a subcone of C. By $\mathrm{bd}(C)$ we denote the set $C \setminus \mathrm{int}(C)$.

Definition 1.12 Let C_a and C_b be two salient half-spaces. Their *direct sum* is the salient half-space $C_a \oplus C_b$, consisting of all ordered pairs $x = (x^a, x^b)$ with $x^a \in C_a$ and $x^b \in C_b$. The salient half-space operations are for all $x, y \in C_a \oplus C_b$ and for all $\alpha \geq 0$ given by:

$$\begin{cases} (x+y)^a & := & x^a + y^a \\ (\alpha x)^a & := & \alpha x^a \end{cases} \quad \text{and} \quad \begin{cases} (x+y)^b & := & x^b + y^b \\ (\alpha x)^b & := & \alpha x^b. \end{cases}$$

For every $x \in C_a \oplus C_b$, there are unique $x^a \in C_a$ and $x^b \in C_b$ such that $x = (x^a, x^b)$. Since $C_a \oplus C_b$ is a salient half-space, every property derived for salient half-spaces is also applicable to $C_a \oplus C_b$.

On the direct sum $C_a \oplus C_b$ the partial order relation $\leq_{C_a \oplus C_b}$ satisfies:

$$x \ \leq_{C_a \oplus C_b} \ y \Longleftrightarrow \begin{cases} x^a \leq_{C_a} y^a \\ x^b \leq_{C_b} y^b. \end{cases}$$

We conclude this short introduction to direct sums by remarking that

$$V[C_a \oplus C_b] = V[C_a] \oplus V[C_b],$$

where the second $\oplus$ denotes the usual direct sum defined for two vector spaces (cf. [4]). In the following, we present the definition of a cone-dependent set of a salient half-space C and state its relationship with the definition of a linear dependent set in $V[C]$. We conlcude this subsection by stating that every salient half-space C has a maximal cone-independent set and that this set is a basis for the vector space reproduced by C.

Definition 1.13 Let S be a subset of a salient half-space C. Then S is *cone-dependent* if $0 \in S$ or if there is a non-empty, finite subset U of S such that

$$\operatorname{cone}(U) \cap \operatorname{cone}(S \setminus U) \neq \{0\}.$$

The set S is *cone-independent* if S is not cone-dependent, i.e., if $0 \notin S$ and every non-empty, finite subset U of S satisfies

$$\operatorname{cone}(U) \cap \operatorname{cone}(S \setminus U) = \{0\}.$$

Lemma 1.14 *Let S be a subset of a salient halfspace C. Then*

$$S \textit{ is cone-dependent } \iff S \textit{ is linearly dependent in } V[C].$$

Proof

The above lemma is obviously true in case $0 \in S$, hence, throughout this proof, we assume $0 \notin S$.

If S is cone-dependent, there is $x \in C \setminus \{0\}$ and there is a non-empty, finite set $U \subset S$ such that $x \in \operatorname{cone}(U) \cap \operatorname{cone}(S \setminus U)$. Clearly, $x \in \operatorname{span}_{V[C]}(U) \cap \operatorname{span}_{V[C]}(S \setminus U)$, and so S is linearly dependent in $V[C]$.

For the converse, assume S is linearly dependent in $V[C]$. Then $\exists n \in I\!N \ \exists s_1, \ldots, s_n \in S$, satisfying $s_i \neq s_j \ (i \neq j) \ \exists \lambda_1, \ldots \lambda_n \in I\!R \setminus \{0\} : \sum_{i=1}^{n} \lambda_i s_i = 0$. Since C is pointed, there is $k \in I\!N$ such that $1 < k < n$ and $\forall i \in \{1, \ldots, k\} : \lambda_i < 0$ and $\forall i \in \{k+1, \ldots, n\} : \lambda_i > 0$. Now, if $x = \sum_{i=1}^{k} (-\lambda_i) s_i$ then $x \neq 0$ and $x = \sum_{i=k+1}^{n} \lambda_i s_i$, i.e., $x \neq 0$ and $x \in \operatorname{cone}(\{s_1, \ldots, s_k\}) \cap \operatorname{cone}(\{s_{k+1}, \ldots, s_n\})$. $\square$

Corollary 1.15 *Let S be a subset of a salient halfspace C. Then*

$$S \text{ is cone-independent } \iff S \text{ is linearly independent in } V[C].$$

For every salient half-space C, the family $\mathcal{M}$ of cone-independent sets can be partially ordered by inclusion: for all $S_1, S_2 \in \mathcal{M}$ define $S_1 \leq S_2 \;:\iff\; S_1 \subseteq S_2$. A totally ordered family or chain is a partially ordered set such that every two elements of the set are comparable. An upper bound of a subset $\mathcal{N} \subset \mathcal{M}$ is an element $U \in \mathcal{M}$ such that $\forall S \in \mathcal{N} : S \leq U$. A maximal element of $\mathcal{M}$ is an element $U \in \mathcal{M}$ such that $\forall S \in \mathcal{M} : \; U \leq S \implies U = S$.

Proposition 1.16 *Let C be a salient half-space. Then C has a maximal cone-independent subset.*

Proof
To prove the proposition, we use Zorn's Lemma. So, let $\mathcal{K}$ be a chain in the family $\mathcal{M}$ of cone-independent subsets of C. We show that $\sup(\mathcal{K}) := \bigcup_{S \in \mathcal{K}} S$ is an upper bound for this chain. Clearly $\forall S \in \mathcal{K} : S \subseteq \sup(\mathcal{K})$, so we only have to prove that $\sup(\mathcal{K})$ is cone-independent. Let U be a non-empty, finite subset of $\sup(\mathcal{K})$. Let $x \in \mathrm{cone}(U) \cap \mathrm{cone}(\sup(\mathcal{K}) \backslash U)$, then there is a finite set $V \subset \sup(\mathcal{K}) \backslash U$ such that $x \in \mathrm{cone}(U) \cap \mathrm{cone}(V)$. Since $\mathcal{K}$ is a chain, there is a $S_0 \in \mathcal{K}$ such that $U \cup V \subseteq S_0$. Since $U \cap V = \emptyset$ we find $x \in \mathrm{cone}(U) \cap \mathrm{cone}(S_0 \setminus U)$ and since S_0 is cone-independent, we conclude $x = 0$. $\square$

For two maximal cone-independent sets $S_{\max}$ and $\tilde{S}_{\max}$, each $x \in S_{\max}$ can be associated with a finite subset of $\tilde{S}_{\max}$. So, the cardinality of $S_{\max}$ is not greater than the cardinality of $\tilde{S}_{\max}$. Interchanging the role of $S_{\max}$ and $\tilde{S}_{\max}$, we find that they have the same cardinallity.

Definition 1.17 Let C be a salient half space, and let $S_{\max}$ be a maximal cone-independent set of C. The *dimension* of C, denoted by $\dim(C)$, equals the cardinality of $S_{\max}$.

As a result, a salient half-space C is *finite-dimensional* if $S_{\max}$ is finite.

Lemma 1.18 *Every maximal cone-independent set $S_{\max}$ in a salient half-space C, satisfies* $\mathrm{span}_{V[C]}(S_{\max}) = V[C]$.

Proof
Since $S_{\max}$ is maximal, we find that for every $x \in C \setminus \{0\} : S_{\max} \cup \{x\}$ is cone-dependent, i.e., for every $x \in C \setminus \{0\}$ there is a finite subset $U \subseteq S_{\max}$ such that $\mathrm{cone}(U \cup \{x\}) \cap \mathrm{cone}(S_{\max} \setminus U) \neq \{0\}$. So, $\exists u \in \mathrm{cone}(U)\ \exists v \in \mathrm{cone}(S_{\max} \setminus U)$ such that $x + u = v$. Hence, $[(x, 0)] = [(v, u)]$, which is an element of $\mathrm{span}_{V[C]}(S_{\max})$. $\square$

Corollary 1.15 and Lemma 1.18 imply that for a salient half-space C, every maximal cone-independent set is a basis for $V[C]$. However, in general, $S_{\max}$ is too small to fully describe C; clearly, $\mathrm{cone}(S_{\max}) = C$ does not have to hold, since this would imply that every cone in a finite-dimensional vector space is finitely generated.

Half-dual space

Let C be a salient half-space, as defined in the previous subsection.

Definition 1.19 A positive functional $p : C \to \mathbb{R}^+$ is *half-linear* if p satisfies

$$\begin{cases} p(x+y) = p(x) + p(y) & \forall x, y \in C \\ p(\alpha x) = \alpha p(x) & \forall x \in C \ \forall \alpha \geq 0. \end{cases}$$

The set of all positive half-linear functionals defined on C will be denoted by C^*. If in C^*, addition and positive scalar multiplication are defined pointwise, then C^* is a salient half-space with the zero-functional as its vertex. We call C^* the *half-dual space* of C.

Note that for a direct sum $C_a \oplus C_b$ of two salient half-spaces

$$(C_a \oplus C_b)^* = C_a^* \oplus C_b^*,$$

where the action of $p \in C_a^* \oplus C_b^*$ on an element $x \in C_a \oplus C_b$ is defined by

$$[x, p]_{C_a \oplus C_b} \quad = \quad [x^a, p^a]_{C_a} + [x^b, p^b]_{C_b}.$$

It turns out that existence of one order unit in a salient half-space C is sufficient to guarantee that the half-dual space C^* is non-trivial, i.e., $C^* \neq \{0\}$.

Proposition 1.20 *If C has an order unit, then $C^* \neq \{0\}$.*

Proof
Let u be an order unit for C. Define the set $U \subset V[C]$ by $U := \{\lambda[(u,0)] \mid \lambda \in \mathbb{R}\}$, then U is a subspace of $V[C]$. By Lemma 1.8, we find

$$\forall [(y_1, y_2)] \in V[C] \ \exists \lambda \geq 0 : -\lambda[(u,0)] \leq [(y_1, y_2)] \leq \lambda[(u,0)].$$

Thus, we can define the sublinear functional $q : V[C] \to \mathbb{R}$ by

$$q([(y_1, y_2)]) := \inf\{\lambda \mid [(y_1, y_2)] \leq \lambda[(u,0)]\}.$$

Define $f(\lambda[(u,0)]) := \lambda$, for every $\lambda \in \mathbb{R}$. With this definition, $f : U \to \mathbb{R}$ is a positive linear functional on U satisfying $\forall \lambda \in \mathbb{R} : f(\lambda[(u,0)]) = q(\lambda[(u,0)])$. By the Hahn-Banach Theorem, there exists a linear functional $\tilde{f} : V[C] \to \mathbb{R}$ such that on the set U, $\tilde{f}$ is equal to f, and $\forall [(y_1, y_2)] \in V[C] : \tilde{f}([(y_1, y_2)]) \leq q([(y_1, y_2)])$. For every $[(x_1, x_2)] \in C$ it holds that $q([(x_1, x_2)]) \geq 0$. We conclude that the functional $\tilde{f}$ acts positively on C since for all $[(x_1, x_2)] \in C : \tilde{f}(-[(x_1, x_2)]) \leq q(-[(x_1, x_2)]) \leq 0$. $\square$

The partial order relation $\leq_{C^*}$ on C^* satisfies

$$\begin{aligned} p \leq_{C^*} q &\iff \forall x \in C : p(x) \leq q(x). \\ p <_{C^*} q &\iff (\forall x \in C : p(x) \leq q(x)) \wedge (\exists x \in C : p(x) < q(x)). \end{aligned}$$

Besides a partial order relation on $V[C^*]$ (cf. Definition 1.6 and subsequent construction), the partial order relation $\leq_{C^*}$ on C^* also induces a partial order relation $\leq_*$ on $(V[C])^*$:

$$\begin{aligned} f \leq_* g &\iff \forall x \in C : f(x) \leq g(x). \\ f <_* g &\iff (\forall x \in C : f(x) \leq g(x)) \wedge (\exists x \in C : f(x) < g(x)). \end{aligned}$$

Next, we examine the relationship between the vector space $V[C^*]$, reproduced by the half-dual C^* of C, and the dual space $(V[C])^*$ of $V[C]$.

Proposition 1.21 *$V[C^*]$ is canonically, linearly injected in $(V[C])^*$. Furthermore, $C^* = \{p \in (V[C])^* \mid \forall x \in C : p(x) \geq 0\}$.*

Proof
Let $[(p_1, p_2)] \in V[C^*]$ and define for every $[(y_1, y_2)] \in V[C]$:

$$[(p_1, p_2)]\,([(y_1, y_2)]) := p_1(y_1) - p_1(y_2) - p_2(y_1) + p_2(y_2).$$

It is easy to check that this definition is independent of the choice of the representatives (y_1, y_2) and (p_1, p_2), and that with this definition $[(p_1, p_2)]$ acts as a linear functional on $V[C]$. Secondly, it is easy to check that the mapping, described above, which associates a linear functional to every pair $[(p_1, p_2)] \in V[C^*]$ is linear. Furthermore, this mapping is injective: if $\forall [(x_1, x_2)] \in V[C]$ it holds that $[(p_1, p_2)]([(x_1, x_2)]) = 0$, then $\forall x \in C$: $[(p_1, p_2)]([(x, 0)]) = p_1(x) - p_2(x) = 0$, and we conclude $p_1 = p_2$, or, in other words, $[(p_1, p_2)] = [(0, 0)]$. $\square$

In the sequel we shall regard C^* and $V[C^*]$ as a subset of $(V[C])^*$.

We recall that for a vector space W, a set $S \subset W^*$ is said to be separating the elements of a subset $M \subset W$ if $\forall x, y \in M, x \neq y\ \exists f \in S : f(x) \neq f(y)$. If M is linear, this comes down to $\forall x \in M \setminus \{0\}\ \exists f \in S : f(x) \neq 0$.

Lemma 1.22 *A set $S \subset C^*$ separates the elements of C if and only if the collection $S_{V[C]} := \{[(p_1, p_2)] \in V[C^*] \mid p_1, p_2 \in S\}$ separates the elements of $V[C]$.*

Proof
Let $x, y \in C$. Consider the following sequence of equivalent statements

$$\begin{aligned}
&\forall p \in S : p(x) = p(y),\\
&\forall p_1, p_2 \in S : p_1(x) + p_2(y) = p_1(y) + p_2(x),\\
&\forall [(p_1, p_2)] \in S_{V[C]} : p_1(x) + p_2(y) - p_1(y) - p_2(x) = 0,\\
&\forall [(p_1, p_2)] \in S_{V[C]} : [(p_1, p_2)]\,([(x, y)]) = 0.
\end{aligned}$$

Note that $x \neq y$ is equivalent with $[(x, y)] \neq [(0, 0)]$. $\square$

For each $x \in C$ there is a natural action of x as a positive half-linear functional ϵ_x on C^*. So, the set $\{\epsilon_x \mid x \in C\}$ separates the elements of C^*. The canonical mapping $x \to \epsilon_x$ from C into C^{**}, defined by $\epsilon_x(p) := p(x)$, $p \in C^*$, is injective and linear. With $C^{**} = C$ we mean that the canonical injection from C to C^{**} is also surjective, i.e., all positive functionals on C^* arrise from elements of C, or, in symbols: $C^{**} = \{\epsilon_x \mid x \in C\}$.

Definition 1.23 A salient half-space C is *reflexive* if $C^{**} = C$.

Clearly, if a salient half-space C is reflexive, then C^* separates the elements of C. Using Lemma 1.22, this yields that for a reflexive salient half-space C, the vector space $V[C^*]$ separates the elements of $V[C]$. Furthermore, if C^* separates the elements of C and if C is finite-dimensional, then

$$V[C^*] = (V[C])^*.$$

Conversely, if C is a finite-dimensional salient half-space for which C^* does not separate the elements of C, then, by Lemma 1.22, $V[C^*]$ does not separate $V[C]$, hence $V[C^*] \neq (V[C])^*$. So, if C is a finite-dimensional salient half-space satisfying $V[C^*] = (V[C])^*$, then C^* separates the elements of C. Hence, we find the following lemma.

Lemma 1.24 *Let C be a finite-dimensional salient half-space. Then*

$$C^* \text{ separates the elements of } C \iff V[C^*] = (V[C])^*.$$

Note that, in general, it is not true, that $V[C^*] = (V[C])^*$ implies that C is reflexive, since reflexivity is related to a topological condition on C (cf. Corollary 1.33).

If C is reflexive, then we like to identify each $x \in C$ with the functional $\epsilon_x \in C^{**}$. Since in this situation also C^* is reflexive, we introduce the notation $[x,p]$ for every $p(x)$ and $\epsilon_x(p)$, $x \in C, p \in C^*$. Note, that if C is a reflexive salient half-space, then $\dim(C) = \dim(C^{**})$.

If C is a reflexive salient half-space then $x \leq_C y$ is equivalent with $\forall q \in C^* : [x,q] \leq [y,q]$. Indeed, let $x, y \in C$, then

$$\begin{aligned} x \leq_C y &\iff \exists z \in C : x + z = y \\ &\iff \exists z \in C^{**} : x + z = y \\ &\iff \forall q \in C^* : [q,x] \leq [q,y] \\ &\iff \forall q \in C^* : [x,q] \leq [y,q]. \end{aligned}$$

Topology

Let C be a salient half-space. If $\| \, . \, \|$ is a norm on the vector space $V[C]$, then $d : C \times C \to I\!R^+$, defined by $d(x,y) := \| \, [(x,y)] \, \|$ is a metric on C. Clearly, the function d, thus defined, is real-valued, finite and non-negative. Furthermore, d is symmetric since $d(x,y) = \| \, [(x,y)] \, \| = \| -[(y,x)] \, \| = \| \, [(y,x)] \, \| = d(y,x)$. Finally, the triangle inequality and $d(x,y) = 0$ if and only if $x = y$ follow from the equivalent properties of the norm.

Conversely, if $d : C \times C \to I\!R^+$ is a metric on C, satisfying

$$\begin{cases} \forall x, y \in C \; \forall \alpha \geq 0 : & d(\alpha x, \alpha y) = \alpha d(x,y), \text{ (homogeneity of degree 1)}, \\ \forall x, y, z \in C : & d(x+z, y+z) = d(x,y), \text{ (translation invariance)}, \end{cases}$$

then by $\| \, [(x_1,x_2)] \, \| := d(x_1,x_2)$ a norm is defined on $V[C]$. Indeed, this norm is defined independently of the choice of representatives: if $[(x_1,x_2)] = [(y_1,y_2)]$, i.e., if $x_1+y_2 = x_2+y_1$, then $\| \, [(x_1,x_2)] \, \| = d(x_1,x_2) = d(x_1+y_1, x_2+y_1) = d(x_1+y_1, x_1+y_2) = d(y_1,y_2) = \| \, [(y_1,y_2)] \, \|$. Furthermore, the norm defined above satisfies the triangle inequality: $\| \, [(x_1,x_2)] + [(y_1,y_2)] \, \| = d(x_1+y_1, x_2+y_2) \leq d(x_1+y_1, x_2+y_1) + d(x_2+y_1, x_2+y_2) = d(x_1,x_2) + d(y_1,y_2) = \| \, [(x_1,x_2)] \, \| + \| \, [(y_1,y_2)] \, \|$.

Recall, that for a metric d on C, a subset S of C is d-bounded if and only if $\exists x_0 \in C \; \exists \alpha > 0 : S \subset \{x \in C \mid d(x_0,x) \leq \alpha\}$, i.e., if and only if $\exists \alpha > 0 \; \forall s,t \in S : d(s,t) \leq \alpha$.

Since we regard the salient half-space, rather than the vector space $V[C]$, to be the essential concept of this paper, we would like to have a salient half-space related introduction of topology on C. Hence, in the following proposition, we present conditions on C^* which guarantee the existence of a metric on C, which is related to an internal point of the half dual-space C^*.

Proposition 1.25 *Let C be a salient half-space for which C^* admits an order unit p_0. Then $d_{p_0} : C \times C \to \mathbb{R}^+$, defined by*

$$d_{p_0}(x, y) := \inf\{[v + w, p_0] \mid v, w \in C \text{ with } x + v = y + w\},$$

is a semi-metric on C (cf. [5]).
For every $x, x_1, y, y_1, z \in C$ and $\alpha, \beta \geq 0$ this semi-metric satisfies

- $d_{p_0}(\alpha x, \alpha y) = \alpha d_{p_0}(x, y)$ *(homogeneity of degree 1),*
- $d_{p_0}(x + z, y + z) = d_{p_0}(x, y)$ *(translation invariance),*
- *if $x + y_1 = x_1 + y$ then $d_{p_0}(x, y) = d_{p_0}(x_1, y_1)$,*
- $d_{p_0}(\alpha x, \beta x) = |\alpha - \beta| d_{p_0}(x, 0)$,
- $d_{p_0}(x, 0) = [x, p_0]$.

If, in addition, C^* separates the elements of C, then d_{p_0} is a metric on C.

Proof
It can be easily checked that d_{p_0} is finite, non-negative and symmetric and satisfies $\forall x \in C : d_{p_0}(x, x) = 0$. So, in order to complete the proof that d_{p_0} is a semi-metric, we prove that the triangle inequality holds. Let $x, y, z \in C$, then

$$\begin{aligned}
&d_{p_0}(x, y) + d_{p_0}(y, z)\\
&= \inf\{[v_1 + w_1, p_0] \mid v_1, w_1 \in C \text{ with } x + v_1 = y + w_1\}\\
&\quad + \inf\{[v_2 + w_2, p_0] \mid v_2, w_2 \in C \text{ with } y + v_2 = z + w_2\}\\
&= \inf\{[v_1 + w_1 + v_2 + w_2, p_0] \mid v_1, v_2, w_1, w_2 \in C,\ x + v_1 = y + w_1 \text{ and } y + v_2 = z + w_2\}\\
&\geq \inf\{[v_1 + w_1 + v_2 + w_2, p_0] \mid v_1, v_2, w_1, w_2 \in C,\ x + v_1 + y + v_2 = y + w_1 + z + w_2\}\\
&= \inf\{[v + w, p_0] \mid v, w \in C \text{ with } x + v = z + w\} = d_{p_0}(x, z).
\end{aligned}$$

Translation invariance and homogeneity of degree 1 are easily checked by the reader. The fact that $x + y_1 = x_1 + y$ implies $d_{p_0}(x, y) = d_{p_0}(x_1, y_1)$, is already proved at the beginning of this subsection. It is not difficult to prove that this, combined with the symmetry of d_{p_0}, implies $\forall x \in C\ \forall \alpha, \beta \geq 0 : d_{p_0}(\alpha x, \beta x) = |\alpha - \beta| d_{p_0}(x, 0)$.

To prove that $\forall x \in C : d_{p_0}(x, 0) = [x, p_0]$, we remark that $\forall x \in C : [x, p_0] \leq d_{p_0}(x, 0)$, since for all $v, w \in C$ satisfying $x + v = w$ it holds that $x \leq_C x + 2v = v + w$. Furthermore, we can choose $v = 0$ and $w = x$ to obtain that $d_{p_0}(x, 0) \leq [x, p_0]$.

Finally, we show that if the half-dual space C^* separates the elements of C, then $d_{p_0}(x, y) = 0$ implies $x = y$. If $d_{p_0}(x, y) = 0$, there are sequences $(v_n)_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ in C such that $\forall n \in \mathbb{N} : x + v_n = y + w_n$, and $\lim_{n\to\infty} [v_n, p_0] = \lim_{n\to\infty} [w_n, p_0] = 0$. Since p_0 is an order unit, we find that $\forall q \in C^* : \lim_{n\to\infty} [v_n, q] = \lim_{n\to\infty} [w_n, q] = 0$. Hence, $\forall q \in C^* : [x, q] - [y, q] = \lim_{n\to\infty} [w_n, q] - \lim_{n\to\infty} [v_n, q] = 0$. Since C^* separates C, we conclude $x = y$. $\square$

The previous proposition implies that for a salient half-space C, every element q of its half-dual space C^* generates a semi-metric d_q on C. If $q_1, q_2 \in C^*$ satisfy $q_1 \leq_{C^*} q_2$, then for all $x, y \in C$ we find $d_{q_1}(x,y) \leq d_{q_2}(x,y)$. Furthermore, in case C^* separates the elements of C, then every element of $\text{int}(C^*)$ generates a metric on C. (Recall that for every reflexive salient half-space C, the half-dual space C^* separates the elements of C.) Since for all $p_0, q_0 \in \text{int}(C^*)$ there are $\mu, \lambda > 0$ such that $\mu p_0 \leq_{C^*} q_0 \leq_{C^*} \lambda p_0$, we conclude that all these metrics, generated by internal points of C^*, are equivalent. We denote the topology on C, generated by any $p_0 \in \text{int}(C^*)$, by $\mathcal{T}(C, \text{int}(C^*))$.

A subset $S \subset C$ is called bounded if S is d_{p_0}-bounded for any $p_0 \in \text{int}(C^*)$. Note that S is bounded if and only if $\exists p_0 \in \text{int}(C^*)\ \exists \alpha > 0\ \forall s \in S : [s, p_0] \leq \alpha$.

On a direct sum $C = C_a \oplus C_b$ of salient half-spaces, where C^* separates the elements of C, every element $p_0 \in \text{int}(C^*) = \text{int}(C_a^*) \oplus \text{int}(C_b^*)$ induces a metric d_{p_0} on C, where d_{p_0} satisfies $\forall x, y \in C : d_{p_0}(x,y) = d_{p_0^a}(x^a, y^a) + d_{p_0^b}(x^b, y^b)$.

As mentioned at the beginning of this subsection, we can relate a norm on $V[C]$ to every metric d_{p_0}, with $p_0 \in \text{int}(C^*)$, by defining

$$\| [(x_1, x_2)] \|_{p_0} := d_{p_0}(x_1, x_2),$$

for every $[(x_1, x_2)] \in V[C]$. Hence, every order unit of C^*, where C is a salient half-space for which C^* separates C, induces a norm on $V[C]$. Note that all these norms are equivalent and therefore all topologies, generated by these norms, are equal. By $\mathcal{T}(V[C], \text{int}(C^*))$, we denote the unique topology on $V[C]$, generated by any order unit of C^*. Regarding C as a subset of $V[C]$, the topology $\mathcal{T}(C, \text{int}(C^*))$ is the relative topology of $\mathcal{T}(V[C], \text{int}(C^*))$. Note that if C is finite-dimensional, the set $\text{int}(C)$, consisting of all internal points of C, coincides with the $\mathcal{T}(V[C], \text{int}(C^*))$-interior of C. Hence, every $\mathcal{T}(V[C], \text{int}(C^*))$-interior point of C is an order unit.

Lemma 1.26 *Let C be a salient half-space for which C^* separates the elements of C, and let $p_0 \in \text{int}(C^*)$. Then*

$$\forall q \in C^*\ \exists \lambda_q > 0\ \forall x, y \in C : |[x, q] - [y, q]| \leq \lambda_q d_{p_0}(x, y).$$

Proof
Since p_0 is an order unit for C^*, we find $\forall q \in C^*\ \exists \lambda_q \geq 0 : q \leq_{C^*} \lambda_q p_0$. Hence, for all $x, y, v, w \in C$, satisfying $x + v = y + w$, and for all $q \in C^*$ we find

$$|\, [x, q] - [y, q] \,| \ \leq\ [v, q] + [w, q] \ \leq\ \lambda_q([v, p_0] + [w, p_0]).$$

Taking the infimum of $[v + w, p_0]$ over $v, w \in C$ we find

$$\forall q \in C^*\ \forall x, y \in C :\ |\, [x, q] - [y, q] \,| \leq\ \lambda_q d_{p_0}(x, y).$$

$\square$

Corollary 1.27 *Let C be a salient half-space for which C^* separates C, and for which $\text{int}(C^*) \neq \emptyset$. Then every $q \in C^*$ is a continuous positive half-linear functional on C, with respect to toplogy $\mathcal{T}(C, \text{int}(C^*))$.*

The following proposition follows directly from Lemma 1.26 and the fact that every linear functional $[(p_1, p_2)] \in V[C^*]$ can be written as $[(p_1, p_2)] = [(p_1, 0)] - [(p_2, 0)]$.

Proposition 1.28 *Let C be a salient half-space for which C^* separates C, and for which $\mathrm{int}(C^*) \neq \emptyset$. Then every linear functional $[(p_1, p_2)] \in V[C^*]$ is continuous with respect to $\mathcal{T}(V[C], \mathrm{int}(C^*))$.*

Proposition 1.29 *Let X be an infinite-dimensional normed vector space. Then there is an unbounded linear functional $\mathcal{F} : X \to \mathbb{R}$.*

Proof
Let $H = \{h_i \mid i \in \mathbb{N}\}$ be a maximal linearly independent subset, or Hamel basis (cf. [2]), in X, such that $\forall i \in \mathbb{N} : \| h_i \| = 1$, and for every $x \in X$, let the function $\mathcal{H}_x : H \to \mathbb{R}$ be defined by $x = \sum_{i \in \mathbb{N}} \mathcal{H}_x(h_i) h_i$. Now define the linear functional $\mathcal{F} : X \to \mathbb{R}$ by

$$\mathcal{F}(x) := \sum_{i \in \mathbb{N}} i \mathcal{H}_x(h_i).$$

Since for every $i \in \mathbb{N}$ we find $\mathcal{F}(h_i) = i$, we conclude that $\mathcal{F}$ is unbounded. $\square$

Let C be a salient half-space for which C^* separates C, and for which $\mathrm{int}(C^*) \neq \emptyset$, and $V[C^*] = (V[C])^*$. Then every linear functional on $V[C]$ is continuous with respect to $\mathcal{T}(V[C], \mathrm{int}(C^*))$, hence $V[C]$, and therefore also C, is finite-dimensional.
Conversely, at the end of the previous subsection, we have seen that if C is a finite-dimensional salient half-space for which C^* separates C, then $V[C^*] = (V[C])^*$. Hence, we find the following corollary.

Corollary 1.30 *Let C be a salient half-space for which C^* separates C. Then*

$$\left.\begin{array}{l} V[C^*] = (V[C])^* \\ \mathrm{int}(C^*) \neq \emptyset \end{array}\right\} \iff C \text{ is finite-dimensional.}$$

Lemma 1.31 *Let C be a reflexive salient half-space. Let $p_0 \in \mathrm{int}(C^*)$ and let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in C with respect to metric d_{p_0}. Then*

$$\exists x \in C \ \forall q \in C^* : \lim_{n \to \infty} [x_n, q] = [x, q].$$

Proof
By Lemma 1.26, for all $q \in C^*$, $([x_n, q])_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^+$. Define the function $\mathcal{F} : C^* \to \mathbb{R}^+$ by $\mathcal{F}(q) := \lim_{n \to \infty} [x_n, q]$. Since C is reflexive, $\exists x \in C \ \forall q \in C^* :$ $\mathcal{F}(q) = [x, q]$. $\square$

Corollary 1.32 *Let C be a finite-dimensional reflexive salient half-space. Then the metric space C is complete.*

Proof
On the one hand, given a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in C, with metric d_{p_0}, there is $y \in V[C]$ such that $\lim_{n \to \infty} \| x_n - y \|_{p_0} = 0$. On the other hand, $\exists x \in C \ \forall q \in C^* : [x, q] = \lim_{n \to \infty} [x_n, q]$. We conclude $x = y$. $\square$

Corollary 1.33 *Let C be a finite-dimensional salient half-space for which C^* separates the elements of C. Then C is reflexive if and only if C is closed in $V[C]$.*

In the remainder of this paper, we use a finite-dimensional reflexive salient half-space to model the set of all economy bundles. In this section, we have seen that every finite-dimensional reflexive salient half-space is a complete metric space, where the metric is generated by any element of the (non-empty) interior of the half-dual space. The following statements will be needed in the proof of the main theorem of this paper.

Corollary 1.34 *Let C be a finite-dimensional, reflexive salient half-space.*

a) *Let $p_0 \in \mathrm{int}(C^*)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C. Then $(x_n)_{n\in\mathbb{N}}$ converges to 0 with respect to the relative topology $\mathcal{T}(C, \mathrm{int}(C^*))$ if and only if $\lim\limits_{n\to\infty}[x_n, p_0] = 0$.*

b) *Let S be a subset of C and let $p_0 \in \mathrm{int}(C^*)$. Then S is bounded if and only if the set $\{[x, p_0] \mid x \in S\}$ is bounded.*

c) *For all $p_0 \in \mathrm{int}(C^*)$, the sets $\{x \in C \mid [x, p_0] \leq 1\}$ and $\{x \in C \mid [x, p_0] = 1\}$ are compact.*

Lemma 1.35 *Let C be a salient half-space for which C^* separates C and $\mathrm{int}(C^*) \neq \emptyset$, let S be a subset of C, and let $u_0 \in \mathrm{int}(C)$. Then S is bounded if $\exists \lambda \geq 0 : S \subset \{x \in C \mid x \leq_C \lambda u_0\}$. If, in addition, C is reflexive and finite-dimensional, then boundedness of S implies $S \subset \{x \in C \mid x \leq_C \lambda u_0\}$, for some $\lambda \geq 0$.*

Proof
Suppose $\exists \lambda \geq 0\ \forall x \in S : x \leq_C \lambda u_0$. Let $p_0 \in \mathrm{int}(C^*)$, then $\forall x \in S : [x, p_0] \leq \lambda[u_0, p_0]$, hence S is bounded.
Now, suppose $\forall \lambda \geq 0\ \exists x \in S : \neg(x \leq_C \lambda u_0)$, i.e., $\forall \lambda \geq 0\ \exists x \in S\ \exists p \in \{q \in C^* \mid [u_0, q] = 1\} : [x, p] > \lambda[u_0, p]$. Then, for every $n \in \mathbb{N}$ there is $x_n \in S$ and $p_n \in \{q \in C^* \mid [u_0, q] = 1\}$ such that $\frac{1}{n}[x_n, p_n] > [u_0, p_n]$. To prove the lemma, we show that the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded. Suppose $(x_n)_{n\in\mathbb{N}}$ is bounded. Since C is assumed to be finite-dimensional and reflexive, we may assume that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with limit $x \in C$, and the sequence $(p_n)_{n\in\mathbb{N}}$ is convergent with limit $p \in \{q \in C^* \mid [u_0, q] = 1\}$. This implies $0 = \lim\limits_{n\to\infty} \frac{1}{n}[x_n, p_n] \geq 1$. We conclude that S is unbounded. □

Next, we present a salient half-space related characterisation of $\mathrm{int}(C)$.

Lemma 1.36 *Let C be a finite-dimensional salient half-space and let $x \in C$. Then $x \in \mathrm{int}(C)$ if and only if $\forall q \in C^* \setminus \{0\} : [x, q] > 0$.*

Proof
Let $x \in \mathrm{int}(C)$ and let $q \in C^* \setminus \{0\}$. There is $y \in C$ with $[y, q] > 0$. Since x is internal point of C, there is $\varepsilon > 0$ and $z \in C$ such that $x = \varepsilon y + z$. We conclude $[x, q] > 0$.
For the converse, suppose $x \in \mathrm{bd}(C) \setminus \{0\}$. Since C is a convex cone, $\mathrm{int}(C)$ is a convex set. By the Weak Separation Theorem of Minkowski ([8, p.60])

$$\exists\, p \in (V[C])^* \setminus \{0\}\ \exists\, \alpha \in \mathbb{R} : \begin{cases} \forall\, \lambda \geq 0 : & p(\lambda x) \leq \alpha \\ \forall\, u_0 \in \mathrm{int}(C) : & p(u_0) \geq \alpha. \end{cases}$$

Choosing λ equal to 0, and choosing a sequence in $\text{int}(C)$ converging to 0, we find $\alpha = 0$. As a consequence $p \in C^* \setminus \{0\}$. By subsequently choosing λ equal to 1, we find $[x,p] \leq 0$. $\square$

Note, that as a consequence of this lemma, we find $(\text{int}(C))^* = C^*$.

Let C be a finite-dimensional, reflexive salient half-space and let $x_0 \in \text{int}(C)$. Then by Corollary 1.34.c, the set $L := \{p \in C^* \mid [x_0, p] = 1\}$ is compact. When we define $\mathcal{U}_{x_0} : C \to I\!R^+$ and $\mathcal{L}_{x_0} : C \to I\!R^+$ by

$$\begin{aligned} \mathcal{U}_{x_0}(x) &:= \max\{[x,p] \mid p \in L\} \\ \mathcal{L}_{x_0}(x) &:= \min\{[x,p] \mid p \in L\}, \end{aligned}$$

then $\mathcal{L}_{x_0}(x) \leq [x,p] \leq \mathcal{U}_{x_0}(x)$ for all $p \in L$ and $x \in C$. Clearly, $\mathcal{L}_{x_0}(x) > 0$ if $x \in \text{int}(C)$. Note, that this also proves that every $x_0 \in \text{int}(C)$ is an order unit.

Lemma 1.37 *Let C be a finite-dimensional, reflexive salient half-space, and let $(x_n)_{n\in I\!N}$ be a sequence in* $\text{int}(C)$*, with limit $x_0 \in \text{int}(C)$. Then, the functions $\mathcal{U}_{x_0} : C \to I\!R^+$ and $\mathcal{L}_{x_0} : C \to I\!R^+$ satisfy*

$$\forall n \in I\!N : \mathcal{L}_{x_0}(x_n)x_0 \leq_C x_n \leq_C \mathcal{U}_{x_0}(x_n)x_0 \text{ and } \lim_{n\to\infty} \mathcal{L}_{x_0}(x_n) = \lim_{n\to\infty} \mathcal{U}_{x_0}(x_n) = 1.$$

Proof
Using the definition of L, $\mathcal{U}_{x_0}$ and $\mathcal{L}_{x_0}$, given above, let $p \in L$ satisfy $[x_0, p] = \mathcal{U}_{x_0}(x_0) = 1$ and, similarly, for all $n \in I\!N$, let $p_n \in L$ satisfy $[x_n, p_n] = \mathcal{U}_{x_0}(x_n)$. Since, for all $n \in I\!N : \mathcal{U}_{x_0}(x_n) \geq [x_n, p]$, we find that $\liminf_{n\to\infty} \mathcal{U}_{x_0}(x_n) \geq [x_0, p] = 1$. Let $(x_{nk})_{k\in I\!N}$ be a subsequence of $(x_n)_{n\in I\!N}$, satisfying $\limsup_{n\to\infty} \mathcal{U}_{x_0}(x_n) = \lim_{k\to\infty} \mathcal{U}_{x_0}(x_{nk})$. The sequence $(p_{nk})_{k\in I\!N}$ lies in the compact set L, so $(p_{nk})_{k\in I\!N}$ can be assumed convergent with limit $q \in L$. Now, we find

$$\limsup_{n\to\infty} \mathcal{U}_{x_0}(x_n) = \lim_{k\to\infty} \mathcal{U}_{x_0}(x_{nk}) = \lim_{k\to\infty} [x_{nk}, p_{nk}] = [x_0, q] = 1 \leq \liminf_{n\to\infty} \mathcal{U}_{x_0}(x_n).$$

A similar argument can be used to prove $\lim_{n\to\infty} \mathcal{L}_{x_0}(x_n) = 1$. $\square$

The proof of the main theorem of this paper makes use of a cone-related fixed point theorem, which is a corollary of the well known fixed point theorem of Brouwer.

Brouwer's Fixed Point Theorem ([2, Theorem V.9.1])
Let K be a non-empty compact convex subset of a finite-dimensional normed vector space X and let $\mathcal{F} : K \to K$ be a continuous function, then there exists $x \in K$ such that $\mathcal{F}(x) = x$, i.e., $\mathcal{F}$ has a fixed point in K.

Brouwer's Fixed Point Theorem has the following consequence for continuous functions on a salient half-space.

Proposition 1.38 *Let C be a salient half-space satisfying $V[C]$ is finite-dimensional and $C^{**} = C$. Let $\mathcal{G} : C \setminus \{0\} \to C$ be a continuous function. Then there exists an $x \in C \setminus \{0\}$ such that $\mathcal{G}(x) = \alpha x$ for some $\alpha \geq 0$. In fact, for all $p_0 \in$* $\text{int}(C^*)$ *there is $x \in C$ such that $\mathcal{G}(x) = [\mathcal{G}(x), p_0]x$.*

Proof
Let $p_0 \in \text{int}(C^*)$. The set $L_1(p_0) := \{x \in C \mid [x, p_0] = 1\}$ is non-empty, convex, and compact by Corollary 1.34.c. Define the function $\mathcal{F} : L_1(p_0) \to L_1(p_0)$ by

$$\mathcal{F}(x) := \frac{x + \mathcal{G}(x)}{1 + [\mathcal{G}(x), p_0]}.$$

Then $\mathcal{F}$ is a continuous function. By the preceding theorem the function $\mathcal{F}$ has a fixed point x in $L_1(p_0)$, so $x = \mathcal{F}(x) = \frac{x+\mathcal{G}(x)}{1+[\mathcal{G}(x),p_0]}$. □

2 Introduction of the model

As mentioned in the introduction, all aspects of the model of a private ownership economy, described in this paper, resemble those of the model presented in [11], except for a slight alteration in the definition of production technology and the fact that we assume that the preference relations of the agents are defined only on the set of consumption bundles.

A recapitulation of the model of [11] with these adaptions, and the definition of an equilibrium concept analogous to the concept of Walrasian equilibrium in this setting, are the main items of this section.

Economy bundles and pricing functions

Both this model and the model of a private ownership economy, presented in [11], differ from the neo-classical models in the following two aspects.

- The model recognises production and consumption as two different features in an economy. Thus, two different types of economy bundles occur: production bundles and consumption bundles. Production bundles are used as input for production processes. The output of a production process is always a consumption bundle. Agents have an initial endowment which may contain bundles of both types, their preference relation, however, is defined only on the set of consumption bundles.

- Also, we follow the idea of [9], where a mathematical model of a pure exchange economy is presented in which commodities are not assumed to occur separately. Instead of introducing the commodity space $(\mathbb{R}^n)^+$ describing n different commodities, only appearance of so called economy bundles is assumed. Here, the term "economy bundle" is used to describe exchangable objects in the economy. Thus, economy bundles can represent single commodities, bundles of commodities or fixed combinations of commodities.

The above described situation is incorporated as follows.
Firstly, considering economy bundles instead of separate commodities, we model the set of all economy bundles by a salient half-space C, reflecting that the only possible manipulations with these bundles are adding and scaling over $\mathbb{R}^+$. If $x, y \in C$ represent two economy bundles then we can speak of the sum $x + y$ of x and y, and if $\alpha \geq 0$ we can speak of the scaled version αx of x. Both $x + y$ and αx are economy bundles in C. Requiring the economy bundle set C to be salient (Condition 1.1.a) describes the fact that it is impossible for two economy bundles to cancel each other out after addition.
Secondly, considering two types of economy bundles, production and consumption bundles, we assume that C is the direct sum of two salient half-spaces C_{prod} and C_{cons}, where C_{prod} consists of all production bundles and C_{cons} consists of all consumption bundles. Both C_{prod} and C_{cons} are assumed to be non-trivial, i.e., assumed to be not equal to $\{0^{\text{prod}}\}$ and $\{0^{\text{cons}}\}$, where 0^{prod} and 0^{cons} denote the vertex of C_{prod} and C_{cons}, respectively. So, C is also non-trivial. In every economy bundle $x \in C$, each of the two types is uniquely represented: $x = (x^{\text{prod}}, x^{\text{cons}})$ with $x^{\text{prod}} \in C_{\text{prod}}$ and $x^{\text{cons}} \in C_{\text{cons}}$.
At the moment, no further mathematical conditions are imposed on the salient half-space C.

Since in our model commodities are not assumed to occur separately, the price of a single commodity is not a meaningful concept. Instead, we speak of the value of an economy bundle, which will be determined on the basis of "pricing functions". These pricing functions are described by positive half-linear functionals on C. The set of all such functionals has been introduced in Section 1 as the half-dual space C^* and we have seen that $C^* = C^*_{\text{prod}} \oplus C^*_{\text{cons}}$. So, for $x \in C$ and $p \in C^*$, the value $\mathcal{V}(x,p)$ of economy bundle x with respect to the pricing function p equals

$$\mathcal{V}(x,p) := [x,p] = [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}} + [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}}.$$

Economic agents

The features of an economic agent are an economy bundle $w = (w^{\text{prod}}, w^{\text{cons}}) \in C$, modelling his initial endowment, a preference relation $\succeq$ defined on C_{cons}, modelling his taste, and share rates, to be defined in the subsection titled "Equilibrium". By $x^{\text{cons}} \succeq y^{\text{cons}}$ we denote that the agent considers consumption bundle x^{cons} to be at least as preferable as bundle y^{cons}. By $x^{\text{cons}} \succ y^{\text{cons}}$ we mean $x^{\text{cons}} \succeq y^{\text{cons}}$ and $\neg(y^{\text{cons}} \succeq x^{\text{cons}})$. This preference relation $\succeq$ on C_{cons} satisfies reflexivity, transitivity and completeness. The budget set

$$B(p^{\text{cons}}, \kappa) := \{x^{\text{cons}} \in C_{\text{cons}} \mid [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} \leq \kappa\},$$

consists of all consumption bundles that can be afforded at a given value $\kappa \geq 0$ and given pricing function $p^{\text{cons}} \in C^*_{\text{cons}}$.

There are I agents in the economy, indexed by $i \in \{1, \ldots, I\}$, with initial endowment $w_i \in C$ and preference relation $\succeq_i$ defined on C_{cons}. The set

$$D_i(p^{\text{cons}}, \kappa_i) := \{x^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i) \mid \forall y^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i) : x^{\text{cons}} \succeq_i y^{\text{cons}}\}$$

is called the demand set of agent i at given value κ_i and pricing function p^{cons}. Later on, for each agent i the value, or income, κ_i at pricing function $p \in C^*$ will be defined as being the value $\mathcal{V}(w_i, p)$ of his initial endowment plus his shares in the profits of production.

Production processes and technologies

Since we deal with an exchange economy with production, we have to model so called production processes which we regard as processes that incorporate the possibility of converting production bundles into consumption bundles. In our model, the production process that converts production bundle $x^{\text{prod}} \in C_{\text{prod}}$ into consumption bundle $x^{\text{cons}} \in C_{\text{cons}}$, is uniquely represented by the economy bundle $x = (x^{\text{prod}}, x^{\text{cons}}) \in C$.
A production technology $T \subset C$ will be defined as a collection of production processes, that satisfies

- the production process "no production" belongs to T,
- a production process in T with zero input has zero output,
- the free disposal properties.

The following concepts, related to a direct sum of salient half-spaces, are needed in the formal definition of production technology.

Definition 2.1 For all $x \in C_{\text{prod}} \oplus C_{\text{cons}}$ the set F_x is given by

$$F_x := \{z \in C \mid x^{\text{prod}} \leq_{\text{prod}} z^{\text{prod}} \text{ and } z^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}}\}.$$

Let $T \subset C$. For all $x \in T$ the set $R_x(T)$ is given by

$$R_x(T) := \{z \in T \mid x \in F_z \text{ and } F_z \subset T\}.$$

Furthermore, the set $E(T)$ is given by

$$E(T) := \{e \in T \mid R_e(T) = \{e\}\}.$$

The following three properties immediately follow from the above definition.

Lemma 2.2 *Let $x \in C$. Then*

- $x \in R_x(T) \iff F_x \subset T.$
- $\forall y \in F_x : F_y \subset F_x.$
- $y \in F_x$ *and* $x \in F_y \quad \iff \quad x = y.$

Note that the set F_x is closely connected with the notion of "free disposal of commodities". Later, we show that the set $E(T)$ describes the set of all efficient elements of a set T of production processes.
First, we give the definition of a production technology. Note that it is different from the definition used in [11].

Definition 2.3 A set $T \subset C$ is a *production technology* if

a) $(0^{\text{prod}}, 0^{\text{cons}}) \in T$,

b) If $(0^{\text{prod}}, x^{\text{cons}}) \in T$ then $x^{\text{cons}} = 0^{\text{cons}}$,

c) $T = \bigcup_{e \in E(T)} F_e.$

A production process $(x^{\text{prod}}, x^{\text{cons}})$ in a production technlogy T is said to be efficient, if at least x^{prod} is needed to produce x^{cons}, and if it is not possible to produce more than x^{cons} out of x^{prod}. Mathematically speaking, this boils down to the following definition.

Definition 2.4 For a production technology T, a production process $e \in T$ is *efficient* if $\forall x \in T$:

$$\left.\begin{array}{l} x^{\text{prod}} \leq_{\text{prod}} e^{\text{prod}} \\ e^{\text{cons}} \leq_{\text{cons}} x^{\text{cons}} \end{array}\right\} \implies x = e.$$

Put differently, $e \in T$ is efficient if and only if $e \in E(T)$ (cf. Definition 2.1). Note that $(0^{\text{prod}}, 0^{\text{cons}}) \in E(T)$.

Given a pricing function $p \in C^*$ and a production process $x \in C$, we define the profit or gain $\mathcal{G}(x,p)$ by

$$\mathcal{G}(x,p) := [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} - [x^{\text{prod}}, p^{\text{prod}}]_{\text{prod}}.$$

Note that the following two properties are a direct consequence of the definition of $\mathcal{G}$ and F_x.

- Let $x \in C$, $p \in C^*$ and $y \in F_x$, then $\mathcal{G}(x,p) \geq \mathcal{G}(y,p)$.
- Let $x \in C$, $p \in \mathrm{int}(C^*)$ and let $y \in F_x$ satisfy $y \neq x$, then $\mathcal{G}(x,p) > \mathcal{G}(y,p)$.

So, for every pair $(x,p) \in C \times C^*$ we can speak both of its value, where x is considered as an economy bundle, and of its gain, where x is considered as a production process. Note that $\mathcal{V}(x,p) \in \mathbb{R}^+$ and $\mathcal{G}(x,p) \in \mathbb{R}$.

There are J production technologies in the economy, denoted by T_j, $j \in \{1,\dots,J\}$. Given $p \in C^*$, the (possibly empty) set of all gain maximizing production processes in T_j is called the supply set $S_j(p)$ of T_j, i.e.,

$$S_j(p) = \{x \in T_j \mid \forall y \in T_j : \mathcal{G}(x,p) \geq \mathcal{G}(y,p)\}.$$

Equilibrium

For agent i, $i \in \{1,\dots,I\}$, $\theta_{ij}\mathcal{G}(x_j,p)$ is the share in the profit of production technology T_j, $j \in \{1,\dots,J\}$, when production process $x_j \in T_j$ is executed at pricing function p. For all $i \in \{1,\dots,I\}$ and $j \in \{1,\dots,J\}$ the share rates satisfy $\theta_{ij} \geq 0$. Furthermore, $\sum_{i=1}^{I} \theta_{ij} = 1$, for all $j \in \{1,\dots,J\}$.
Given pricing function $p \in C^*$ and production processes $x_j \in T_j$ for all $j \in \{1,\dots,J\}$, the value $\kappa_i(p;x_1,\dots,x_J)$ of agent i, $i \in \{1,\dots,I\}$, is

$$\kappa_i(p;x_1,\dots,x_J) := \mathcal{V}(w_i,p) + \sum_{j=1}^{J} \theta_{ij}\mathcal{G}(x_j,p).$$

In this setting, an equilibrium concept analogous to that of the neo-classical Walrasian equilibrium can be introduced.

Definition 2.5 An element $p_{\mathrm{eq}} \in C^*$ is a *(Walrasian) equilibrium pricing function* if $\forall j \in \{1,\dots,J\}\ \exists s_j \in S_j(p_{eq})\ \forall i \in \{1,\dots,I\}\ \exists d_i^{\mathrm{cons}} \in D_i(p_{eq}, \kappa_i(p_{eq}; s_1,\dots,s_J))$:

$$\sum_{i=1}^{I}(0^{\mathrm{prod}}, d_i^{\mathrm{cons}}) + \sum_{j=1}^{J}(s_j^{\mathrm{prod}}, 0^{\mathrm{cons}}) \leq_C \sum_{i=1}^{I} w_i + \sum_{j=1}^{J}(0^{\mathrm{prod}}, s_j^{\mathrm{cons}}),$$

i.e.,

$$\begin{cases} \sum_{j=1}^{J} s_j^{\mathrm{prod}} & \leq_{\mathrm{prod}} & \sum_{i=1}^{I} w_i^{\mathrm{prod}}, \\ \sum_{i=1}^{I} d_i^{\mathrm{cons}} & \leq_{\mathrm{cons}} & \sum_{i=1}^{I} w_i^{\mathrm{cons}} + \sum_{j=1}^{J} s_j^{\mathrm{cons}}. \end{cases}$$

3 From production technology to supply

In this section, we derive some properties of production technologies, and introduce some new related items. Thereafter, we show that under Assumption 3.6, stated below, for each $j \in \{1, \ldots, J\}$ and for every pricing function p, taken from some specific set Domain[j] $\subset C^*$ (c.f. Definition 3.4), the supply set $S_j(p) = \{x \in T_j \mid \forall y \in T_j : \mathcal{G}(x,p) \geq \mathcal{G}(y,p)\}$ consists of exactly one element. Furthermore, we show that Assumption 3.6 implies that the supply function $\mathcal{S}_j$, defined on the set Domain[j], such that $S_j(p) = \{\mathcal{S}_j(p)\}$ for all $p \in$ Domain[j], is continuous. Finally, for each $j \in \{1, \ldots, J\}$, the behaviour of the supply function $\mathcal{S}_j$ is investigated regarding a sequence $(p_n)_{n \in I\!N} \in$ Domain[j] with limit $p \notin$ Domain[j].

Before we make several assumptions for every production technology, note that the following lemma is a direct consequence of the definition of production technology and of the continuity of the order relations $\geq_{\text{prod}}$ and $\geq_{\text{cons}}$.

Lemma 3.1 *Let T_j be a production technology and let $(x_n)_{n \in I\!N}$ be a convergent sequence in T_j, with limit $x \in C$. Let $(e_n)_{n \in I\!N}$ be a sequence in $E(T_j)$ satisfying $\forall n \in I\!N : x_n \in F_{e_n}$. If $(e_n)_{n \in I\!N}$ is convergent with limit $e \in E(T_j)$, then $x \in F_e \subset T_j$.*

The following lemma implies that if a production technology T_j satisfies $\forall e, f \in E(T_j)\ \forall \tau \in [0,1] : \tau e + (1-\tau) f \in T_j$, then T_j is convex.

Lemma 3.2 *Let S be a subset of the salient half-space $C = C_{\text{prod}} \oplus C_{\text{cons}}$. Assume $S = \bigcup_{e \in E(S)} F_e$ and assume $\forall e, f \in E(S)\ \forall \tau \in [0,1] : \tau e + (1-\tau) f \in S$. Then the set S is convex.*

Proof

Let $x, y \in S$ and $\tau \in [0,1]$. By the first property of S, there exist $e, f \in E(S)$ such that $x \in F_e$ and $y \in F_f$. Thus,

$$\begin{cases} \exists \tilde{x}^{\text{prod}} \in C_{\text{prod}} : x^{\text{prod}} = e^{\text{prod}} + \tilde{x}^{\text{prod}} \\ \exists \tilde{x}^{\text{cons}} \in C_{\text{cons}} : e^{\text{cons}} = x^{\text{cons}} + \tilde{x}^{\text{cons}} \end{cases} \quad \text{and} \quad \begin{cases} \exists \tilde{y}^{\text{prod}} \in C_{\text{prod}} : y^{\text{prod}} = f^{\text{prod}} + \tilde{y}^{\text{prod}} \\ \exists \tilde{y}^{\text{cons}} \in C_{\text{cons}} : f^{\text{cons}} = y^{\text{cons}} + \tilde{y}^{\text{cons}}. \end{cases}$$

To prove convexity of S we shall show that $\tau x + (1-\tau) y \in F_{(\tau e + (1-\tau) f)}$. Indeed, this proves the assertion since both properties of S, combined with the second property of Lemma 2.2, yield $F_{(\tau e + (1-\tau) f)} \subset S$.
Firstly, note that

$$\begin{aligned} \tau x^{\text{prod}} + (1-\tau) y^{\text{prod}} &= \tau (e^{\text{prod}} + \tilde{x}^{\text{prod}}) + (1-\tau)(f^{\text{prod}} + \tilde{y}^{\text{prod}}) \\ &= (\tau e^{\text{prod}} + (1-\tau) f^{\text{prod}}) + (\tau \tilde{x}^{\text{prod}} + (1-\tau) \tilde{y}^{\text{prod}}), \end{aligned}$$

and secondly,

$$(\tau x^{\text{cons}} + (1-\tau) y^{\text{cons}}) + (\tau \tilde{x}^{\text{cons}} + (1-\tau) \tilde{y}^{\text{cons}}) = \tau e^{\text{cons}} + (1-\tau) f^{\text{cons}}.$$

Since $\tau \tilde{x}^{\text{prod}} + (1-\tau) \tilde{y}^{\text{prod}} \in C_{\text{prod}}$ and $\tau \tilde{x}^{\text{cons}} + (1-\tau) \tilde{y}^{\text{cons}} \in C_{\text{cons}}$, we conclude that $\tau x + (1-\tau) y \in F_{(\tau e + (1-\tau) f)}$. $\square$

In order to define the above mentioned sets Domain[j], $j \in \{1, \ldots, J\}$, we need the following definition.

Definition 3.3 For every production technology T_j, $j \in \{1, \ldots, J\}$, the extended real function $\chi_j : C^* \to [0, \infty]$ is given by

$$\chi_j(p) := \sup_{x \in T_j} \mathcal{G}(x, p) = \sup_{e \in E(T_j)} \mathcal{G}(e, p).$$

Note that for every $j \in \{1, \ldots, J\}$, the function χ_j is convex, i.e.,

$$\forall p_1, p_2 \in C^* \; \forall \tau \in [0, 1] : \chi_j(\tau p_1 + (1 - \tau)p_2) \leq \tau \chi_j(p_1) + (1 - \tau)\chi_j(p_2).$$

Definition 3.4 For every $j \in \{1, \ldots, J\}$ the set Domain$[j]$ is given by

$$\text{Domain}[j] := \{q \in C^* \setminus \{0\} \mid \exists x_q \in T_j : \mathcal{G}(x_q, q) = \chi_j(q)\}.$$

With this definition, the supply set $S_j(q)$ is, for every $q \in \text{Domain}[j]$, is given by

$$S_j(q) = \{x \in T_j \mid \mathcal{G}(x, q) = \chi_j(q)\}.$$

For every $q \notin \text{Domain}[j]$ we find $S_j(q) = \emptyset$.
Note that $\forall j \in \{1, \ldots, J\} \; \forall q \in \text{Domain}[j] : S_j(q) \cap E(T_j) \neq \emptyset$.

In order to state the assumptions concerning production technology T_j, $j \in \{1, \ldots, J\}$, which are needed in the proof of the main theorem of this paper, we need a topology on the salient half-space C. Furthermore, since some of the statements in this section make use of the compactness of bounded, closed subsets of C, we make the following assumption, which will be one of the conditions of the main theorem of this paper.

Assumption 3.5 *The salient half-space C, used in the model of a private ownership economy of Section 2 to model the set of all economy bundels, is finite-dimensional and reflexive.*

Under this assumption, C can be endowed with the topology $\mathcal{T}(C, \text{int}(C^*))$, defined in Section 1. For the remainder of this section, let d be a metric on C, corresponding with topology $\mathcal{T}(C, \text{int}(C^*))$.

Next, we lay assumptions on the production technologies T_j, $j \in \{1, \ldots, J\}$. These assumptions imply that we can deal with continuous supply functions $\mathcal{S}_j : \text{Domain}[j] \to C$, and will be part of the conditions of the main theorem.

Assumption 3.6 *Every production technology $T_j \subset C$, $j \in \{1, \ldots, J\}$ satisfies*

a) *if $e_1, e_2 \in E(T_j)$, $e_1 \neq e_2$, $\tau \in (0, 1)$ then $\tau e_1 + (1 - \tau)e_2 \in \text{int}(T_j)$,*

b) *$E(T_j)$ is closed with respect to the topology $\mathcal{T}(C, \text{int}(C^*))$.*

Note that these assumptions on the production technologies of our model are different from the corresponding ones concerning production technologies in [11]. Firstly, part a is related to the concept of internal point, introduced in Section 1, where the former version was not. Secondly, part b only requires closedness of $E(T_j)$ instead of closedness of T_j. The connection between closedness of T_j and $E(T_j)$ is investigated below. Also, note that under these assumptions, every production technology is convex.

Lemma 3.7 *Let T_j be a production technology, let $a^{\text{prod}} \in C_{\text{prod}}$ and let C_{cons} satisfy $\text{int}(C_{\text{cons}}) \neq \emptyset$. If T_j is closed, then the set $\{x^{\text{cons}} \in C_{\text{cons}} \mid (a^{\text{prod}}, x^{\text{cons}}) \in T_j\}$ is bounded.*

Proof

Let $b_0 \in \text{int}(C_{\text{cons}})$. Suppose the set $\{x^{\text{cons}} \in C_{\text{cons}} \mid (a^{\text{prod}}, x^{\text{cons}}) \in T_j\}$ is unbounded, then, by Lemma 1.35, for every $n \in I\!N$ there exists $x_n^{\text{cons}} \in C_{\text{cons}}$ such that

$$\begin{cases} (a^{\text{prod}}, x_n^{\text{cons}}) \in T_j \\ x_n^{\text{cons}} \geq_{\text{cons}} n b_0^{\text{cons}}. \end{cases}$$

By Definition 2.3.c we find $(a^{\text{prod}}, n b_0^{\text{cons}}) \in T_j$ for all $n \in I\!N$. Since T_j is convex and contains $(0^{\text{prod}}, 0^{\text{cons}})$ (Definition 2.3.a), we find $\forall n \in I\!N : (\frac{1}{n} a^{\text{prod}}, b_0^{\text{cons}}) \in T_j$. Taking the limit for $n \to \infty$, the clossedness of T_j implies $(0^{\text{prod}}, b_0^{\text{cons}}) \in T_j$, which is in contradiction with Definition 2.3.b. $\square$

Lemma 3.8 *Let T_j be a production technology and let $S \subset T_j$ satisfy $\exists a^{\text{prod}} \in C_{\text{prod}}\ \forall x \in S : x^{\text{prod}} \leq_{\text{prod}} a^{\text{prod}}$. If T_j is closed, then S is bounded.*

Proof

Let $x \in S$. By Definition 2.3.c, we find that $x^{\text{prod}} \leq_{\text{prod}} a^{\text{prod}}$ implies $(a^{\text{prod}}, x^{\text{cons}}) \in F_x \subset T_j$, so $S \subset \{y \in C \mid (a^{\text{prod}}, y^{\text{cons}}) \in T_j\}$. By the previous lemma we find that S is bounded. $\square$

Lemma 3.9 *Let T_j be a production technology, satisfying $E(T_j)$ is closed, and assume every sequence $(e_n)_{n \in I\!N}$ in $E(T_j)$ satisfies*

$$(e_n^{\text{prod}})_{n \in I\!N} \text{ is bounded} \quad \Longrightarrow \quad (e_n)_{n \in I\!N} \text{ is bounded.}$$

Then T_j is closed.

Proof

Let $(x_n)_{n \in I\!N}$ be a convergent sequence in T_j with limit $x \in C$. By Definition 2.3.c we find a sequence $(e_n)_{n \in I\!N}$ in $E(T_j)$ satisfying $\forall n \in I\!N : x_n \in F_{e_n}$. Hence, $\forall n \in I\!N : x_n^{\text{prod}} \geq_{\text{prod}} e_n^{\text{prod}}$. Since the sequence $(e_n^{\text{prod}})_{n \in I\!N}$ is bounded, the assumption implies that $(e_n)_{n \in I\!N}$ is bounded. Without loss of generality, we may assume that $(e_n)_{n \in I\!N}$ is convergent with limit $e \in E(T_j)$. By the continuity of the order relations $\geq_{\text{prod}}$ and $\geq_{\text{cons}}$, and by Definition 2.3.c we find $x \in F_e \subset T_j$. $\square$

In case $\text{int}(C_{\text{cons}}) \neq \emptyset$, the previous three lemmas imply that for a production technology T_j satisfying $E(T_j)$ is closed, the following two statements are equivalent:

- T_j is closed,
- "bounded input yields bounded output".

Next, we show that Assumptions 3.5 and 3.6, indeed imply that we can deal with continuous supply functions $\mathcal{S}_j$, defined on the set Domain$[j]$, $j \in \{1, \ldots, J\}$. For the remainder of this section, let j be any fixed element of $\{1, \ldots, J\}$, and assume Domain$[j] \neq \emptyset$. Before we are able to define the supply function $\mathcal{S}_j$, we need uniqueness of the supply, for every $p \in$ Domain$[j]$. This, combined with some properties of the unique element of the supply set, is proved in the following lemma.

Lemma 3.10 *Let $p \in \mathrm{Domain}[j]$. Then there is a unique $e_p \in E(T_j)$ such that $\mathcal{G}(e_p, p) = \chi_j(p)$. Furthermore, e_p satisfies $\forall x \in S_j(p) : x \in F_{e_p}$. Finally, if $p \in \mathrm{Domain}[j] \cap \mathrm{int}(C^*)$, then e_p is the unique element of supply set $S_j(p)$.*

Proof

Since $p \in \mathrm{Domain}[j]$, the set $S_j(p) \cap E(T_j)$ is non-empty. Suppose $e_1, e_2 \in S_j(p) \cap E(T_j)$ and $e_1 \neq e_2$. Then by Assumption 3.6.a, $x := \tau e_1 + (1-\tau)e_2$ is an internal point of T_j. Hence, for a fixed order unit u_0 of C there exists $\varepsilon > 0$ such that $(x^{\mathrm{prod}}, x^{\mathrm{cons}} + \varepsilon u_0^{\mathrm{cons}}) \in (\varepsilon u_0^{\mathrm{prod}}, 0^{\mathrm{cons}}) + T_j$. Let $y \in T_j$ satisfy $(\varepsilon u_0^{\mathrm{prod}}, 0^{\mathrm{cons}}) + y = (x^{\mathrm{prod}}, x^{\mathrm{cons}} + \varepsilon u_0^{\mathrm{cons}})$. Since $p \neq 0$, we find $\mathcal{G}(y,p) > \mathcal{G}(x,p)$ which is in contradiction with the optimality of e_1 and e_2. We conclude that there is a unique $e_p \in S_j(p) \cap E(T_j)$, maximising $\mathcal{G}(e,p)$, $e \in E(T_j)$. Furthermore, $\forall x \in S_j(p) : x \in F_{e_p}$.
Let $p \in \mathrm{Domain}[j] \cap \mathrm{int}(C^*)$ and let $x \in T_j \setminus E(T_j)$. Then $\exists e_x \in E(T_j) : x \in F_{e_x}$. Since $p \in \mathrm{int}(C^*)$ and $x \neq e_x$ we find $\mathcal{G}(x,p) < \mathcal{G}(e_x,p) \leq \mathcal{G}(e_p,p)$. □

Definition 3.11 The *supply function* $\mathcal{S}_j : \mathrm{Domain}[j] \to E(T_j)$ is given by $S_j(q) \cap E(T_j) = \{\mathcal{S}_j(q)\}$, for all $q \in \mathrm{Domain}[j]$

Lemma 3.12 *Let $(p_n)_{n \in \mathbb{N}}$ be a sequence in $\mathrm{Domain}[j]$, with limit $p \neq 0$. If the sequence $(\mathcal{S}_j(p_n))_{n \in \mathbb{N}}$ is convergent with limit $s \in C$, then $p \in \mathrm{Domain}[j]$ and $s = \mathcal{S}_j(p)$.*

Proof

Since $\forall n \in \mathbb{N}\ \forall x \in T_j : \mathcal{G}(\mathcal{S}_j(p_n), p_n) \geq \mathcal{G}(x, p_n)$, continuity of $\mathcal{G}$ guarantees $\forall x \in T_j$: $\mathcal{G}(s,p) \geq \mathcal{G}(x,p)$. Since $E(T_j)$ is closed (Assumption 3.6.b), $s \in E(T_j)$, so $p \in \mathrm{Domain}[j]$. Furthermore, Lemma 3.10 implies $s = \mathcal{S}_j(p)$. □

Lemma 3.13 *The supply function $\mathcal{S}_j : \mathrm{Domain}[j] \to E(T_j)$ is continuous with respect to the relative topology on $\mathrm{Domain}[j]$.*

Proof

Let $(p_n)_{n \in \mathbb{N}}$ be a sequence in $\mathrm{Domain}[j]$, with limit $p \in \mathrm{Domain}[j]$. Suppose $(\mathcal{S}_j(p_n))_{n \in \mathbb{N}}$ does not converge to $\mathcal{S}_j(p)$. Without loss of generality, we may assume $\exists \varepsilon > 0\ \ \forall n \in \mathbb{N}$: $d(\mathcal{S}_j(p_n), \mathcal{S}_j(p)) > \varepsilon$. Define $x_n := \tau_n \mathcal{S}_j(p_n) + (1-\tau_n)\mathcal{S}_j(p)$, with $\tau_n := \frac{\varepsilon}{d(\mathcal{S}_j(p_n), \mathcal{S}_j(p))} \in (0,1)$. Then $d(x_n, \mathcal{S}_j(p)) = \varepsilon$ and by Assumption 3.6.a we find that x_n is an internal point of T_j. Both the sequences $(\tau_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ are bounded. Without loss of generality assume $\lim\limits_{n\to\infty} \tau_n = \tau$ and $\lim\limits_{n\to\infty} x_n = x \in C$. Note that $x \neq \mathcal{S}_j(p)$ implies $\tau > 0$. Since $\mathcal{G}(x_n, p_n) \geq \min\{\mathcal{G}(\mathcal{S}_j(p_n), p_n), \mathcal{G}(\mathcal{S}_j(p), p_n)\} = \mathcal{G}(\mathcal{S}_j(p), p_n)$, the continuity of $\mathcal{G}$ implies $\mathcal{G}(x,p) \geq \mathcal{G}(\mathcal{S}_j(p), p)$. Since $\forall n \in \mathbb{N} : \mathcal{G}(x_n, p) \leq \chi_j(p)$, we find $\mathcal{G}(x,p) = \chi_j(p)$.
Since for all $n \in \mathbb{N} : x_n = \tau_n \mathcal{S}_j(p_n) + (1-\tau_n)\mathcal{S}_j(p)$, the sequence $(\mathcal{S}_j(p_n))_{n \in \mathbb{N}}$ in $E(T_j)$ is convergent with limit $e \in E(T_j)$ satisfying $x = \tau e + (1-\tau)\mathcal{S}_j(p))$. Note that $x \neq \mathcal{S}_j(p)$ implies $e \neq \mathcal{S}_j(p)$. However, $\mathcal{G}(e,p) = \frac{1}{\tau}(\mathcal{G}(x,p) - (1-\tau)\mathcal{G}(\mathcal{S}_j(p), p)) = \chi_j(p)$. This is in contradiction with $\mathcal{S}_j(p)$ being the unique element of the set $S_j(p) \cap E(T_j)$. □

Corollary 3.14 *The function χ_j is continuous on $\mathrm{Domain}[j]$.*

The continuity of the supply function is proved, so we can now concentrate on some other properties of this function. First, we derive some limit behaviour, especially regarding a sequence $(p_n)_{n\in\mathbb{N}} \in \mathrm{Domain}[j]$, with limit $p \notin \mathrm{Domain}[j]$. Also, we will investigate the set $\mathrm{Domain}[j]$ in more detail.

Lemma 3.15 *Let C_{prod} satisfy $\mathrm{int}(C_{\mathrm{prod}}) \neq \emptyset$, let $p_0 \in \mathrm{int}(C^*)$, let $\alpha \in \mathbb{R}$, and let $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ be an unbounded subset of C. Then the set $\{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$ is unbounded.*

Proof
Let $u_0^{\mathrm{prod}} \in \mathrm{int}(C_{\mathrm{prod}})$. Then, by the free-disposal property of T_j, for every $y \in \{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ there is $\lambda > 0$ such that $(y^{\mathrm{prod}} + \lambda u_0^{\mathrm{prod}}, y^{\mathrm{cons}}) \in \{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$. $\square$

Lemma 3.16 *Let $p_0 \in \mathrm{int}(C^*)$, let $\alpha \in \mathbb{R}$ satisfy $\alpha < \chi_j(p_0)$ and let $\{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$ be a bounded set. Then $\chi_j(p_0) < \infty$.*

Proof
Let $(e_n)_{n\in\mathbb{N}}$ be a sequence in $E(T_j)$, satisfying $\sup\limits_{n\in\mathbb{N}} \mathcal{G}(e_n, p_0) = \chi_j(p_0)$. By Lemma 3.15, the set $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is bounded, so $(e_n)_{n\in\mathbb{N}}$ has a convergent subsequence with limit $e \in E(T_j) \cap S_j(p_0)$. $\square$

Corollary 3.17 *Let $p_0 \in \mathrm{int}(C^*)$ and let $\alpha \in \mathbb{R}$. If $\chi_j(p_0) = \infty$ then the set $\{x \in T_j \mid \mathcal{G}(x, p_0) = \alpha\}$ is unbounded.*

Lemma 3.18 *Let $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(C^*)$. Then there is a $\mathcal{T}(C, \mathrm{int}(C^*))$-open neighbourhood O of p_0 such that every $q \in O$ satisfies $\chi_j(q) < \infty$.*

Proof
Let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{int}(C^*)$, converging to p_0, such that $\forall n \in \mathbb{N} : \chi_j(q_n) = \infty$. By the previous corollary, for all $n \in \mathbb{N}$, the set $L_n := \{z \in T_j \mid \mathcal{G}(z, q_n) = \mathcal{G}(\mathcal{S}_j(p_0), q_n)\}$ is unbounded, so $\forall n \in \mathbb{N}\ \exists y_n \in L_n : \mathcal{V}(y_n, p_0) > 1 + \mathcal{V}(\mathcal{S}_j(p_0), p_0)$. Since L_n is convex, and contains $\mathcal{S}_j(p_0)$, for all $\tau \in [0, 1]$ we find $\tau y_n + (1 - \tau)\mathcal{S}_j(p_0) \in L_n$. Now choose $\tau_n := \frac{1}{\mathcal{V}(y_n, p_0) - \mathcal{V}(\mathcal{S}_j(p_0), p_0)} \in (0, 1)$ then $x_n := \tau_n y_n + (1 - \tau_n)\mathcal{S}_j(p_0) \in L_n \cap U$ where $U := \{z \in C \mid \mathcal{V}(z, p_0) = 1 + \mathcal{V}(\mathcal{S}_j(p_0), p_0)\}$. Since U is compact (Corollary 1.34.c), we may as well assume that $(x_n)_{n\in\mathbb{N}}$ is convergent, with limit $x \in C$. Note that the continuity of $\mathcal{G}$ implies $\mathcal{G}(x, p_0) = \chi_j(p_0)$.
By Definition 2.3.c, there is a sequence $(e_n)_{n\in\mathbb{N}}$ in $E(T_j)$ satisfying $\forall n \in \mathbb{N} : x_n \in F_{e_n}$. Hence, $\mathcal{G}(x_n, p_0) \leq \mathcal{G}(e_n, p_0) \leq \chi_j(p_0)$ and $x_n^{\mathrm{prod}} \geq_{\mathrm{prod}} e_n^{\mathrm{prod}}$. So, the sequence $(\mathcal{G}(e_n, p_0))_{n\in\mathbb{N}}$ is convergent with limit $\chi_j(p_0)$, and the sequence $(e_n^{\mathrm{prod}})_{n\in\mathbb{N}}$ is bounded. Moreover, $[e_n^{\mathrm{cons}}, p_0^{\mathrm{cons}}]_{\mathrm{cons}} \leq \chi_j(p_0) + [e_n^{\mathrm{prod}}, p_0^{\mathrm{prod}}]_{\mathrm{prod}}$, so the sequence $(e_n^{\mathrm{cons}})_{n\in\mathbb{N}}$ is bounded. Without loss of generality, we assume that $(e_n)_{n\in\mathbb{N}}$ is convergent. Let $e \in E(T_j)$ be its limit, so $\mathcal{G}(e, p_0) = \chi_j(p_0)$. By Lemma 3.10 we find $e = \mathcal{S}_j(p_0)$. Continuity of $\geq_{\mathrm{prod}}$ and $\geq_{\mathrm{cons}}$ implies $x \in F_e \subset T_j$. Now, $x \in T_j$ and $\mathcal{G}(x, p_0) = \chi_j(p_0)$ imply that x is an element of the supply set $S_j(p_0)$. Since $x \in U$ implies $x \neq \mathcal{S}_j(p_0)$, we arrive at a contradiction since $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(C^*)$ combined with Lemma 3.10 implies that $\mathcal{S}_j(p_0)$ is the unique element of the supply set $S_j(p_0)$. $\square$

Lemma 3.19 *Let $p_0 \in \mathrm{int}(C^*)$, let $\alpha \in I\!R$, and let $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ be an unbounded set. Then $p_0 \notin \mathrm{Domain}[j]$.*

Proof
Let $(x_n)_{n \in I\!N}$ be an unbounded sequence in $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$. Let $0 < \varepsilon \leq 1$ and define $p_\varepsilon := ((1-\varepsilon)p_0^{\mathrm{prod}}, (1+\varepsilon)p_0^{\mathrm{cons}})$. Since for all $n \in I\!N$ the gain $\mathcal{G}(x_n, p_\varepsilon)$ equals $\mathcal{G}(x_n, p_0) + \varepsilon[x_n^{\mathrm{prod}}, p_0^{\mathrm{prod}}]_{\mathrm{prod}} + \varepsilon[x_n^{\mathrm{cons}}, p_0^{\mathrm{cons}}]_{\mathrm{cons}}$, the sequence $(\mathcal{G}(x_n, p_\varepsilon))_{n \in I\!N}$ is unbounded. Hence, $\forall \varepsilon \in (0,1] : \chi_j(p_\varepsilon) = \infty$. Using Lemma 3.18, we conclude $p_0 \notin \mathrm{Domain}[j]$. □

Corollary 3.20 *Let $p_0 \in \mathrm{int}(C^*)$. If $p_0 \in \mathrm{Domain}[j]$, then for all $\alpha \in I\!R$, the set $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is compact.*

Corollary 3.21 *Let $(p_n)_{n \in I\!N}$ be a convergent sequence in $\mathrm{Domain}[j]$, with limit $p \in C^* \setminus \mathrm{Domain}[j]$. Then $\forall p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(C^*) : \limsup_{n \to \infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$.*

Proof
The sequence $(\mathcal{S}_j(p_n))_{n \in I\!N}$ does not have a point of accumulation, since existence of such a point would lead to a contradiction with Lemma 3.12. Let $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(C^*)$. For all $\alpha \in I\!R$, the set $\{x \in T_j \mid \mathcal{G}(x, p_0) \geq \alpha\}$ is compact (Corollary 3.20), and so we find that $\forall \alpha \in I\!R\ \exists N \in I\!N\ \forall n > N : \mathcal{G}(\mathcal{S}_j(p_n), p_0) \leq \alpha$. We conclude $\limsup_{n \to \infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$. □

Proposition 3.22 $\mathrm{Domain}[j] \cap \mathrm{int}(C^*) = \mathrm{int}(\mathrm{Domain}[j])$.

Proof
We only have to prove $\mathrm{Domain}[j] \cap \mathrm{int}(C^*) \subset \mathrm{int}(\mathrm{Domain}[j])$. Let $p_0 \in \mathrm{Domain}[j] \cap \mathrm{int}(C^*)$. By Lemma 3.18, there is a $\mathcal{T}(C, \mathrm{int}(C^*))$-open neighbourhood O of p_0 such that every $q \in O$ satisfies $\chi_j(q) < \infty$. Let $q \in O$. We shall prove that $\exists e \in E(T_j) : \mathcal{G}(e, q) = \chi_j(q)$. Let $(e_n)_{n \in I\!N}$ be a sequence in $E(T_j)$ satisfying $\lim_{n \to \infty} \mathcal{G}(e_n, q) = \chi_j(q) < \infty$. Then, for $\alpha \in I\!R$ chosen sufficiently small, $(e_n)_{n \in I\!N}$ is a sequence in $\{x \in T_j \mid \mathcal{G}(x, q) \geq \alpha\}$. So, by Corollary 3.20, without loss of generality, we may assume $(e_n)_{n \in I\!N}$ is convergent with limit $e \in E(T_j) \subset T_j$ (Assumption 3.6.b). Since $\mathcal{G}(e, q) = \chi_j(q)$, we conclude that $q \in \mathrm{Domain}[j]$. □

Corollary 3.23 *The set $\mathrm{int}(\mathrm{Domain}[j]) \cup \{0\}$ is a subcone of C^*.*

Proof
Since the function $\mathcal{G} : C \times C^* \to I\!R$ is homogeneous of degree one, $\mathrm{Domain}[j] \cup \{0\}$ is closed under scalar multiplication over $I\!R^+$. Let $p_1, p_2 \in \mathrm{int}(\mathrm{Domain}[j])$ and let $\tau \in (0,1)$. We prove that $q := \tau p_1 + (1-\tau) p_2 \in \mathrm{Domain}[j]$. We first note that $p_1, p_2 \in \mathrm{Domain}[j]$ implies $\chi_j(q) \leq \tau \chi_j(p_1) + (1-\tau)\chi_j(p_2)$. Since there is nothing to prove in case $\mathcal{G}(\mathcal{S}_j(p_1), q) = \chi_j(q)$, we may as well assume that $\exists \varepsilon > 0$ such that $\mathcal{G}(\mathcal{S}_j(p_1), q) < \chi_j(q) - \varepsilon$ Define $U := \{x \in T_j \mid \mathcal{G}(x, p_2) \geq \mathcal{G}(\mathcal{S}_j(p_1), p_2)\}$, then U is compact (Lemma 3.20). Let $(e_n)_{n \in I\!N}$ be a sequence in $E(T_j)$ satisfying $\sup_{n \in I\!N} \mathcal{G}(e_n, q) = \chi_j(q)$.

Let $n \in I\!N$. If $e_n \notin U$, i.e., if $\mathcal{G}(e_n, p_2) < \mathcal{G}(\mathcal{S}_j(p_1), p_2)$ then $\mathcal{G}(e_n, q) = \tau\mathcal{G}(e_n, p_1) + (1 - \tau)\mathcal{G}(e_n, p_2) < \tau\mathcal{G}(\mathcal{S}_j(p_1), p_1) + (1-\tau)\mathcal{G}(\mathcal{S}_j(p_1), p_2) = \mathcal{G}(\mathcal{S}_j(p_1), q) < \chi_j(q) - \varepsilon$. We conclude that $\exists N \in I\!N\ \forall n > N : e_n \in U$. Since U is compact, $q \in \mathrm{Domain}[j]$. $\square$

We conclude this section on production technologies and their corresponding supply functions, with the definition of the total supply function, on the set Domain $\subset C^*$, where Domain is defined by

$$\mathrm{Domain} := \bigcap_{j \in \{1,\ldots,J\}} \mathrm{Domain}[j].$$

One of the conditions of the main theorem of this paper, will imply that Domain $\neq \emptyset$.

Definition 3.24 The *total supply function* $\mathcal{S} : \mathrm{Domain} \to C$ is given by

$$\mathcal{S}(p) := \sum_{j=1}^{J} \mathcal{S}_j(p).$$

Corollary 3.25 *Let* $(p_n)_{n \in I\!N}$ *be a sequence in* Domain*, convergent to* $p \in C^* \setminus \mathrm{Domain}$*. Then* $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$ *for all* $p_0 \in \mathrm{Domain} \cap \mathrm{int}(C^*)$*.*

Proof

Let $p_0 \in \mathrm{Domain} \cap \mathrm{int}(C^*)$. For all $j \in \{1, \ldots, J\}$ we find that either $p \in \mathrm{Domain}[j]$ and $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0)$ is finite (Lemma 3.13), or $p \notin \mathrm{Domain}[j]$ and $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}_j(p_n), p_0) = -\infty$ (Corollary 3.21). Since there exists $j_0 \in \{1, \ldots, J\}$ such that $p \notin \mathrm{Domain}[j_0]$, we find $\limsup_{n\to\infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$. $\square$

4 From agent to demand

As mentioned in Section 2, an agent i, $i \in \{1, \dots, I\}$ is characterised by

- initial endowment $w_i = (w_i^{\text{prod}}, w_i^{\text{cons}}) \in C$,
- preference relation $\succeq_i$ defined on C_{cons},
- share rates $\theta_{ij} \in [0, 1]$, $i \in \{1, \dots, I\}$, $j \in \{1, \dots, J\}$, satisfying $\sum_{j=1}^{J} \theta_{ij} = 1$.

In this section, analogous to the situation in the previous section, we introduce assumptions that allow for the definition of demand functions on a suitable domain. Also, the continuity of these demand functions is proved, the limit behaviour of these functions is examined and the total demand function is defined. As mentioned in the introduction, the main difference between this section and the corresponding section of [10], is the restriction of most concepts to the salient half-space C_{cons} of all consumption bundles.

Similar to Assumption 3.6, Assumption 4.1 is stated in a topological setting. Also, some statements of this section will use the fact that every closed and bounded subset of C is compact. Hence, we continue Assumptions 3.5 of the previous section. Recall that under this assumption, the salient half-space C can be endowed with the topology $\mathcal{T}(C, \text{int}(C^*))$, as defined in Section 1. Furthermore, $\text{int}(C) \neq \emptyset$ and for all $p_0 \in \text{int}(C^*)$, the sets $\{x \in C \mid [x, p_0] \leq 1\}$ and $\{x \in C \mid [x, p_0] = 1\}$ are compact.

Assumption 4.1 *For every $i \in \{1, \dots, I\}$, preference relation $\succeq_i$ is*

a) *monotone: $\forall\, x^{\text{cons}}, y^{\text{cons}} \in C_{\text{cons}} : x^{\text{cons}} \leq_{\text{cons}} y^{\text{cons}}$ implies $y^{\text{cons}} \succeq_i x^{\text{cons}}$,*

b) *strictly convex: $\forall x^{\text{cons}}, y^{\text{cons}} \in C_{\text{cons}}$, $\tau \in (0, 1) : x^{\text{cons}} \succeq_i y^{\text{cons}}$ and $x^{\text{cons}} \neq y^{\text{cons}}$ imply $\tau x^{\text{cons}} + (1 - \tau) y^{\text{cons}} \succ_i y^{\text{cons}}$,*

c) *continuous: $\forall y^{\text{cons}} \in C_{\text{cons}}$ the sets $\{x^{\text{cons}} \in C_{\text{cons}} \mid x^{\text{cons}} \succeq_i y^{\text{cons}}\}$ and $\{x^{\text{cons}} \in C_{\text{cons}} \mid y^{\text{cons}} \succeq_i x^{\text{cons}}\}$ are closed in C_{cons}, with respect to topology $\mathcal{T}(C_{\text{cons}}, \text{int}(C^*_{\text{cons}}))$.*

We have seen that for given $\kappa_i \geq 0$, $i \in \{1, \dots, I\}$, and $p^{\text{cons}} \in C^*_{\text{cons}}$, the budget set $B(p^{\text{cons}}, \kappa_i)$ and the demand set $D_i(p^{\text{cons}}, \kappa_i)$ are given by

$$B(p^{\text{cons}}, \kappa_i) := \{x^{\text{cons}} \in C_{\text{cons}} \mid [x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} \leq \kappa_i\},$$

$$D_i(p^{\text{cons}}, \kappa_i) := \{x^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i) \mid \forall y^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i) : x^{\text{cons}} \succeq_i y^{\text{cons}}\}.$$

We start with the derivation of some properties of the budget and the demand sets. These properties will be needed to prove the continuity and the limit behaviour of the demand functions as well as for the proof of Walras' Law. For the remainder of this section, let i be any fixed element of $\{1, \dots, I\}$.

Lemma 4.2 *Let $\kappa_i \geq 0$ and $p^{\text{cons}} \in C^*_{\text{cons}}$. Then demand set $D_i(p^{\text{cons}}, \kappa_i)$ consists of at most one element.*

Proof
This is a direct consequence of Assumption 4.1.b. □

Lemma 4.3 *Let $\kappa_i \geq 0$, $p^{\text{cons}} \in C^*_{\text{cons}}$, $x^{\text{cons}} \in C_{\text{cons}}$ and suppose $\{d^{\text{cons}}\} = D_i(p^{\text{cons}}, \kappa_i)$ with $x^{\text{cons}} >_{\text{cons}} d^{\text{cons}}$. Then $x^{\text{cons}} \notin B(p^{\text{cons}}, \kappa_i)$.*

Proof
Due to the monotony of the preference relation (Assumption 4.1.a), $x^{\text{cons}} >_{\text{cons}} d^{\text{cons}}$ implies $x^{\text{cons}} \succeq_i d^{\text{cons}}$. Since $x^{\text{cons}} \neq d^{\text{cons}}$ and the demand set $D_i(p^{\text{cons}}, \kappa_i)$ consists of at most one element, we arrive at a contradiction. $\square$

Lemma 4.4 *Let $\kappa_i \geq 0$ and let $p^{\text{cons}} \in C^*_{\text{cons}}$. Then $p^{\text{cons}} \in \text{int}(C^*_{\text{cons}}) \iff D_i(p^{\text{cons}}, \kappa_i) \neq \emptyset$.*

Proof
Suppose $p^{\text{cons}} \in \text{int}(C_{\text{cons}})$. By Lemma 1.34.c, the budget set $B(p^{\text{cons}}, \kappa_i)$ is compact in C_{cons}. For every $b^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i)$, define the set $G(b^{\text{cons}}) := \{x^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i) \mid b^{\text{cons}} \succ_i x^{\text{cons}}\}$. The preference relation $\succeq_i$ is continuous (Assumption A4.(c)), so every set $G(b^{\text{cons}})$ is $\mathcal{T}(C, \text{int}(C^*))$-open. Suppose the demand set at value functional p^{cons} and value κ_i were empty, then every element of $B(p^{\text{cons}}, \kappa_i)$ is element of at least one $G(b^{\text{cons}})$. The collection $\{G(b^{\text{cons}}) \mid b^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i)\}$ is an open cover of the compact set $B(p^{\text{cons}}, \kappa_i)$, so there is a finite subset $F \subset B(p^{\text{cons}}, \kappa_i)$ such that $B(p^{\text{cons}}, \kappa_i) = \bigcup_{f^{\text{cons}} \in F} G(f^{\text{cons}})$. The preference relation $\succeq_i$ being transitive, F has a maximal element $f_1^{\text{cons}} \in F$. Since, $f_1^{\text{cons}} \in G(f_2^{\text{cons}})$ for some $f_2^{\text{cons}} \in F$, $f_2^{\text{cons}} \neq f_1^{\text{cons}}$, we arrive at a contradiction.
For the converse, suppose $p^{\text{cons}} \in \text{bd}(C^*_{\text{cons}})$. Then there is an element $x^{\text{cons}} \in C_{\text{cons}} \setminus \{0\}$, such that $[x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = 0^{\text{cons}}$. Since $\forall y^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i) : y^{\text{cons}} + x^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i)$, and since $y^{\text{cons}} + x^{\text{cons}} >_{\text{cons}} y^{\text{cons}}$, Lemma 4.3 yields that $B(p^{\text{cons}}, \kappa_i)$ does not contain a maximal element with respect to $\succeq_i$. $\square$

Lemma 4.5 *Let $\kappa_i \geq 0$, let $p^{\text{cons}} \in \text{int}(C^*_{\text{cons}})$ and let $\{d^{\text{cons}}\} = \mathcal{D}_i(p^{\text{cons}}, \kappa_i)$. Then $[d^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = \kappa_i$.*

Proof
In case $\kappa_i = 0$, the budget set $B(p^{\text{cons}}, \kappa_i)$ equals $\{0^{\text{cons}}\}$, and thus $[d^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = [0^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = 0$. Now, suppose $\kappa_i > 0$ and $[d^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} < \kappa_i$, then there is $x_0^{\text{cons}} \in \text{int}(C_{\text{cons}})$ such that $x_0^{\text{cons}} >_{\text{cons}} d^{\text{cons}}$ and $[x_0^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} > \kappa_i$. Consider the convex combination $\tau x_0^{\text{cons}} + (1-\tau)d^{\text{cons}}$ with $\tau \in (0,1)$ so small that $[\tau x_0^{\text{cons}} + (1-\tau)d^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} \leq \kappa_i$. Then $\tau x_0^{\text{cons}} + (1-\tau)d^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i)$ and $\tau x_0^{\text{cons}} + (1-\tau)d^{\text{cons}} >_{\text{cons}} d^{\text{cons}}$. By Lemma 4.3, we come to a contradiction. $\square$

Lemma 4.6 *Let $p^{\text{cons}} \in C^*_{\text{cons}}$, $\kappa_i > 0$, $x^{\text{cons}} \in C_{\text{cons}}$, and suppose $x^{\text{cons}} \succeq_i b^{\text{cons}}$ for all $b^{\text{cons}} \in C_{\text{cons}}$ satisfying $[b^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} < \kappa_i$. Then $x^{\text{cons}} \succeq_i b^{\text{cons}}$ for all $b^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i)$.*

Proof
We have to prove that $x^{\text{cons}} \succeq_i b^{\text{cons}}$ for all $b^{\text{cons}} \in C_{\text{cons}}$ satisfying $[b^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = \kappa_i$. Clearly, such a b^{cons} satisfies $b^{\text{cons}} \neq 0^{\text{cons}}$. Hence, for all $\tau \in [0,1)$ we have $[\tau b^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} < \kappa_i$ and so $x^{\text{cons}} \succeq_i \tau b^{\text{cons}}$. By Assumption 4.1.c, the preference relation $\succeq_i$ is continuous, so $x^{\text{cons}} \succeq_i b^{\text{cons}}$. $\square$

In Section 2, we already mentioned that the value, or budget, κ_i at pricing function $p \in C^*$ will be the value of the initial endowment of agent i, plus his shares in the profits of production. Since the profits of production are determined by the supply functions, we find the following formal definition of the value function.

Definition 4.7 The *value function* $\mathcal{K}_i : \text{Domain} \to \mathbb{R}_0^+$ is given by

$$\mathcal{K}_i(p) := \mathcal{V}(w_i, p) + \sum_{j=1}^{J} \theta_{ij} \mathcal{G}(\mathcal{S}_j(p), p).$$

For every $p \in \text{Domain}$ the *budget set* $B_i(p)$ and *demand set* $D_i(p)$ are given by

$$B_i(p) := B(p^{\text{cons}}, \mathcal{K}_i(p)),$$

and

$$D_i(p) := D_i(p^{\text{cons}}, \mathcal{K}_i(p)).$$

Lemma 3.13 implies that value function $\mathcal{K}_i$ is continuous on the set Domain.

Let Domain$^\circ$ be defined by

$$\text{Domain}^\circ := \text{Domain} \bigcap \{p \in C^* \mid p^{\text{cons}} \in \text{int}(C^*_{\text{cons}})\}.$$

Note that $\text{Domain}^\circ \cup \{0\}$ is a subcone of C^* and that $\text{int}(\text{Domain}^\circ) = \text{int}(\text{Domain})$.

On the set Domain$^\circ$, we are able to construct the demand function $\mathcal{D}_i$ from the demand sets $D_i(p)$, where $p \in \text{Domain}^\circ$.

Definition 4.8 The *demand function* $\mathcal{D}_i : \text{Domain}^\circ \to C_{\text{cons}}$ is, for all $p \in \text{Domain}^\circ$, given by $D_i(p) = \{\mathcal{D}_i(p)\}$.

We conclude this section on the agents by deriving the continuity and some other properties of the demand functions. Again, let i be any fixed element of $\{1, \ldots, I\}$.

Lemma 4.9 *Let $(p_n)_{n \in \mathbb{N}}$ be a sequence in* Domain$^\circ$ *convergent to $p_0 \in \text{Domain}^\circ$. Then the following two properties hold.*

a) *If $b_n^{\text{cons}} \in B_i(p_n)$ for each $n \in \mathbb{N}$, then there is a subsequence $\left(b_{n_k}^{\text{cons}}\right)_{k \in \mathbb{N}}$ that converges to some $b^{\text{cons}} \in B_i(p_0)$.*

b) *For each $b^{\text{cons}} \in B_i(p_0)$ there exists a convergent sequence $(b_n^{\text{cons}})_{n \in \mathbb{N}}$ with limit b^{cons}, such that $b_n^{\text{cons}} \in B_i(p_n)$ for all $n \in \mathbb{N}$.*

Proof

a) Since $p_0^{\text{cons}} \in \text{int}(C^*_{\text{cons}})$ is an order unit, Lemma 1.37 implies that the function $\mathcal{L}_{p_0^{\text{cons}}} : C^*_{\text{cons}} \to \mathbb{R}^+$ satisfies

$$\lim_{n \to \infty} \mathcal{L}_{p_0^{\text{cons}}}(p_n^{\text{cons}}) = 1 \text{ and } \forall\, n \in \mathbb{N} : \; \mathcal{L}_{p_0^{\text{cons}}}(p_n^{\text{cons}}) p_0^{\text{cons}} \leq_{\text{cons}} p_n^{\text{cons}}.$$

Because $b_n^{\text{cons}} \in B_i(p_n)$ for all $n \in I\!N$, we find

$$\mathcal{L}_{p_0^{\text{cons}}}(p_n^{\text{cons}})[b_n^{\text{cons}}, p_0^{\text{cons}}]_{\text{cons}} \leq [b_n^{\text{cons}}, p_n^{\text{cons}}]_{\text{cons}} \leq \mathcal{K}_i(p_n).$$

Since the function $\mathcal{K}_i : \text{Domain}^\circ \to I\!R^+$ is continuous, the sequence $(\mathcal{K}_i(p_n))_{n\in I\!N}$ is convergent. And since $p_0^{\text{cons}} \in \text{int}(C^*_{\text{cons}})$, boundedness of $[b_n^{\text{cons}}, p_0^{\text{cons}}]_{\text{cons}}$ implies that the sequence $(b_n^{\text{cons}})_{n\in I\!N}$ is bounded (Lemma 1.34.b). So, $(b_n^{\text{cons}})_{n\in I\!N}$ has a convergent subsequence $\left(b_{n_k}^{\text{cons}}\right)_{k\in I\!N}$ with limit $b^{\text{cons}} \in C$. Since for every k in $I\!N$, $\mathcal{L}_{p_0^{\text{cons}}}(p_n^{\text{cons}})[b_{n_k}^{\text{cons}}, p_0^{\text{cons}}]_{\text{cons}} \leq \mathcal{K}_i(p_{n_k})$, the limit b^{cons} belongs to $B_i(p_0)$.

b) Let $b^{\text{cons}} \in B_i(p_0)$. If $[b^{\text{cons}}, p_0^{\text{cons}}]_{\text{cons}} < \mathcal{K}_i(p_0)$ then $\exists N \in I\!N\ \forall n > N : [b^{\text{cons}}, p_n^{\text{cons}}]_{\text{cons}} < \mathcal{K}_i(p_n)$, and so, if we choose $b_n^{\text{cons}} := b^{\text{cons}}$ for all $n > N$, we are done. Therefore, we may as well assume $[b^{\text{cons}}, p_0^{\text{cons}}]_{\text{cons}} = \mathcal{K}_i(p_0)$. For every $n \in I\!N$, define $\tau_n := \frac{\mathcal{K}_i(p_n)}{[b^{\text{cons}}, p_n^{\text{cons}}]_{\text{cons}}}$. Note that $\lim\limits_{n\to\infty} \tau_n = 1$. Now put $b_n^{\text{cons}} := \tau_n b^{\text{cons}}$, then $\forall n \in I\!N$: $[b_n^{\text{cons}}, p_n^{\text{cons}}]_{\text{cons}} = \mathcal{K}_i(p_n)$ and $\lim\limits_{n\to\infty} b_n^{\text{cons}} = b^{\text{cons}}$.

□

Lemma 4.9 expresses the type of continuity that we need in order to prove the continuity of the individual demand function $\mathcal{D}_i$.

Lemma 4.10 *The demand function $\mathcal{D}_i$ is continuous on* Domain°.

Proof
Let $(p_n)_{n\in I\!N}$ be a sequence in Domain° converging to some $p_0 \in$ Domain°. Suppose the sequence $(\mathcal{D}_i(p_n))_{n\in I\!N}$ does not converge to $\mathcal{D}_i(p_0)$, then without loss of generality any subsequence of $(\mathcal{D}_i(p_n))_{n\in I\!N}$ does not converge to $\mathcal{D}_i(p_0)$. By (a) of the preceding lemma, the sequence $(\mathcal{D}_i(p_n))_{n\in I\!N}$ has a subsequence $(\mathcal{D}_i(p_{nk}))_{k\in I\!N}$ that converges to some $b^{\text{cons}} \in B_i(p_0)$. Now, the proof is done if we can show that $b^{\text{cons}} = \mathcal{D}_i(p_0)$. Let $x^{\text{cons}} \in B_i(p_0)$. By (b) of the preceding lemma, for all $n \in I\!N$ there is $x_n^{\text{cons}} \in B_i(p_n)$ satisfying $\lim\limits_{n\to\infty} x_n^{\text{cons}} = x^{\text{cons}}$. Since the preference relation $\succeq_i$ is continuous (Assumption A4.(c)), we find that if $\forall n \in I\!N : \mathcal{D}_i(p_n) \succeq_i x_n^{\text{cons}}$, then $b^{\text{cons}} \succeq_i x^{\text{cons}}$. So, $b^{\text{cons}} = \mathcal{D}_i(p_0)$. □

Next, the continuity of the demand function $\mathcal{D}_i$ being proved, we derive some properties of this function concerning its behaviour regarding a sequence $(p_n)_{n\in I\!N} \in$ Domain° with limit $p \in C^* \setminus \{0\}$.

Lemma 4.11 *Let $(p_n)_{n\in I\!N}$ be a convergent sequence in* Domain° *with limit $p \in C^* \setminus \{0\}$, and assume the sequence $(\mathcal{K}_i(p_n))_{n\in I\!N}$ is convergent with limit $\kappa_i > 0$. If the sequence $(\mathcal{D}_i(p_n))_{n\in I\!N}$ is bounded, then $p^{\text{cons}} \in \text{int}(C^*_{\text{cons}})$.*

Proof
We may as well assume that the sequence $(\mathcal{D}_i(p_n))_{n\in I\!N}$ is convergent with limit d^{cons}. Using Lemma 4.6, we shall prove that $\{d^{\text{cons}}\} = D_i(p^{\text{cons}}, \kappa_i)$. Indeed, let $b^{\text{cons}} \in B(p^{\text{cons}}, \kappa_i)$ satisfy $[b^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} < \kappa_i$. Then there is $N \in I\!N$ such that $\forall\ n > N : [b^{\text{cons}}, p_n^{\text{cons}}]_{\text{cons}} < \mathcal{K}_i(p_n)$. So, $\mathcal{D}_i(p_n) \succeq_i b^{\text{cons}}$ for all $n > N$. Continuity of the preference relation (Assumption 4.1.c) yields $d^{\text{cons}} \succeq_i b^{\text{cons}}$. So, by Lemma 4.6 we conclude that $\{d^{\text{cons}}\} = D_i(p^{\text{cons}}, \kappa_i)$.
□

Corollary 4.12 *A convergent sequence in* Domain$^\circ$ *with limit* $p \in C$, *satisfying* $p^{\text{cons}} \in \text{bd}(C^*_{\text{cons}})$, *and* $\liminf_{n\to\infty} \mathcal{K}_i(p_n) > 0$, *yields an unbounded sequence* $(\mathcal{D}_i(p_n))_{n\in\mathbb{N}}$.

We end this section on agents and their corresponding demand functions by defining the total demand function and stating a property of this function which we will need in the proof of the existence theorem in Section 5.

Definition 4.13 The *total demand function* $\mathcal{D}$: Domain$^\circ \to C_{\text{cons}}$ is given by

$$\mathcal{D}(p) := \sum_{i=1}^{I} \mathcal{D}_i(p).$$

Just like Assumptions 3.5, 3.6 and 4.1, the following assumption will be one of the requirements of the equilibrium existence theorem.

Assumption 4.14 *For every convergent sequence* $(p_n)_{n\in\mathbb{N}}$ *in* Domain$^\circ$ *with limit* $p \in C^*$ *such that* $p^{\text{cons}} \in \text{bd}(C^*_{\text{cons}}) \setminus \{0\}$, *there is* $i_0 \in \{1, \ldots, I\}$ *such that* $\liminf_{n\to\infty} \mathcal{K}_{i_0}(p_n) > 0$.

A combination of the above assumption and Corollary 4.12, has the following consequence for the total demand function.

Corollary 4.15 *Let* $(p_n)_{n\in\mathbb{N}}$ *be a sequence in* Domain$^\circ$ *with limit* $p \in C^*$ *satisfying* $p^{\text{cons}} \in \text{bd}(C^*_{\text{cons}}) \setminus \{0\}$. *Then the sequence* $(\mathcal{D}(p_n))_{n\in\mathbb{N}}$ *is unbounded.*

In Lemma 4.5, we have seen that $\forall i \in \{1, \ldots, I\}$ $\forall p \in$ Domain$^\circ$:

$$[\mathcal{D}_i(p), p^{\text{cons}}]_{\text{cons}} = \mathcal{K}_i(p) = \mathcal{V}(w_i, p) + \sum_{j=1}^{J} \theta_{ij} \mathcal{G}(\mathcal{S}_j(p), p).$$

So, as a consequence of this lemma, we find an adapted version of Walras' law, namely that for all $p \in$ Domain$^\circ$:

$$[\mathcal{D}(p), p^{\text{cons}}]_{\text{cons}} = \mathcal{V}(w_{\text{total}}, p) + \mathcal{G}(\mathcal{S}(p), p), \tag{1}$$

where the total initial endowment $w_{\text{total}} \in C$ is defined by $w_{\text{total}} := \sum_{i=1}^{I} w_i$.

5 Equilibrium

Finally, we come to the main theorem of this paper. The remainder of this section will be devoted to the proof of the following theorem.

Equilibrium Existence Theorem

The model of a private ownership economy, described in Section 2, admits a Walrasian equilibrium pricing function, under the following assumptions:

Assumption A $C = C_{\text{prod}} \oplus C_{\text{cons}}$ *is a finite-dimensional, reflexive salient half-space.*

Assumption B *For every* $j \in \{1, \ldots, J\}$*, production technology* T_j *satisfies*

1) *if* $e_1, e_2 \in E(T_j)$*,* $e_1 \neq e_2$*,* $\tau \in (0,1)$ *then* $\tau e_1 + (1-\tau)e_2 \in \text{int}(T_j)$*,*
2) $E(T_j)$ *is closed with respect to topology* $\mathcal{T}_C$*.*

Assumption C *For every* $i \in \{1, \ldots, I\}$*, preference relation* $\succeq_i$ *is*

1) *monotone:* $\forall\, x^{\text{cons}}, y^{\text{cons}} \in C_{\text{cons}} : x^{\text{cons}} \leq_{\text{cons}} y^{\text{cons}}$ *implies* $y^{\text{cons}} \succeq_i x^{\text{cons}}$*,*
2) *strictly convex:* $\forall x^{\text{cons}}, y^{\text{cons}} \in C_{\text{cons}}$*,* $\tau \in (0,1) : x^{\text{cons}} \succeq_i y^{\text{cons}}$ *and* $x^{\text{cons}} \neq y^{\text{cons}}$ *imply* $\tau x^{\text{cons}} + (1-\tau) y^{\text{cons}} \succ_i y^{\text{cons}}$*,*
3) *continuous:* $\forall y^{\text{cons}} \in C_{\text{cons}}$ *the sets* $\{x^{\text{cons}} \in C \mid x^{\text{cons}} \succeq_i y^{\text{cons}}\}$ *and* $\{x^{\text{cons}} \in C \mid y^{\text{cons}} \succeq_i x^{\text{cons}}\}$ *are closed in* C_{cons}*.*

Assumption D *The set* Domain$^\circ$ *satisfies*

1) $\text{int}(\text{Domain}^\circ) \neq \emptyset$,
2) *for every sequence* $(p_n)_{n \in \mathbb{N}}$ *in* Domain$^\circ$ *with limit in* $C^* \setminus \{0\}$*, there is* $i_0 \in \{1, \ldots, I\}$ *such that* $\liminf\limits_{n \to \infty} \mathcal{K}_{i_0}(p_n) > 0$.

Note that the assumption that C is finite-dimensional, implies that C^* separates the elements of C, and that $\text{int}(C) \neq \emptyset$. As a consequence, C can be endowed with the topology $\mathcal{T}(C, \text{int}(C^*))$, induced by any element $p_0 \in \text{int}(C^*)$. Hence, Assumptions B, C and D, which make use of topological features, are properly stated.

The rather technical Assumption D.2 is related to the minimum income hypothesis. It is obvious that Assumption D.2 is implied by $\sum_{i=1}^{I} w_i \in \text{int}(C)$. However, we will show that this assumption is implied by the following weaker assumption, stating that $\sum_{i=1}^{I} w_i^{\text{prod}}$ is an element of $\text{int}(C_{\text{prod}})$ and that if $(0^{\text{prod}}, p^{\text{cons}}) \in C^* \setminus \{0\}$ adds zero value to the total initial endowment, then there is a production technology which can produce something with positive value.

Assumption 5.1 $\sum\limits_{i=1}^{I} w_i^{\text{prod}} \in \text{int}(C_{\text{prod}})$*, and for all* $p^{\text{cons}} \in C^*_{\text{cons}} \setminus \{0\}$ *satisfying* $\forall i \in \{1, \ldots, I\} : [w_i^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = 0$*, there is* $j_0 \in \{1, \ldots, J\}$ *and* $x \in T_{j_0}$ *such that* $[x^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} > 0$*.*

Lemma 5.2 *Assumption 5.1 implies Assumption D.2 of the Equilibrium Existence Theorem.*

Proof
Let $(p_n)_{n\in I\!N}$ be a sequence in Domain$^\circ$ with limit $p \in C^* \setminus \{0\}$. We have to prove

$$\exists i_0 \in \{1,\dots,I\} : \liminf_{n\to\infty}(\underbrace{[w_{i_0}^{\text{prod}}, p_n^{\text{prod}}]_{\text{prod}}}_{\geq 0} + \underbrace{[w_{i_0}^{\text{cons}}, p_n^{\text{cons}}]_{\text{cons}}}_{\geq 0} + \sum_{j=1}^{J} \theta_{i_0 j} \underbrace{\mathcal{G}(\mathcal{S}_j(p_n), p_n)}_{\geq 0}) > 0.$$

If there exists $i_0 \in \{1,\dots,I\}$ such that $[w_{i_0}, p] > 0$, there is nothing to prove. So, assume $\forall i \in \{1,\dots,I\} : [w_i, p] = 0$. Since $\sum_{i=1}^{I} w_i^{\text{prod}} \in \text{int}(C_{\text{prod}})$, we may as well assume $p^{\text{prod}} = 0^{\text{prod}}$. Note, that this implies $p^{\text{cons}} \neq 0^{\text{cons}}$. Furthermore, we may as well assume that $\forall i \in \{1,\dots,I\} : [w_i^{\text{cons}}, p^{\text{cons}}]_{\text{cons}} = 0$. By Assumption 5.1, $\exists j_0 \in \{1,\dots,J\}\ \exists x \in T_{j_0} : \mathcal{G}(x,p) > 0$. The continuity of the function $\mathcal{G}$ yields $\exists N \in I\!N\ \forall n > N : \mathcal{G}(\mathcal{S}_{j_0}(p_n), p_n) \geq \mathcal{G}(x, p_n) > \frac{1}{2}\mathcal{G}(x,p) > 0$. Take $i_0 \in \{1,\dots,I\}$ such that $\theta_{i_0 j_0} \neq 0$ and the proof is done. $\square$

Before giving the proof of the main theorem, we introduce the continuous function $\mathcal{Z}$ on Domain$^\circ \times C^*$ by

$$\mathcal{Z}(p,q) := [\mathcal{D}(p), q^{\text{cons}}]_{\text{cons}} - \mathcal{G}(\mathcal{S}(p), q) - \mathcal{V}(w_{\text{total}}, q). \tag{2}$$

Walras' law (1) reads

$$\forall p \in \text{Domain}^\circ : \mathcal{Z}(p,p) = 0. \tag{3}$$

In this seting, the equilibrium concept analogous to that of the neo-classical Walrasian equilibrium, introduced in Definition 2.5 can be characterised as follows.

Proposition 5.3 *Let* $p_{\text{eq}} \in \text{Domain}^\circ$*. Then* p_{eq} *is a Walrasian equilibrium pricing function if and only if*

$$(0^{\text{prod}}, \mathcal{D}(p_{\text{eq}})) + (\mathcal{S}(p_{\text{eq}})^{\text{prod}}, 0^{\text{cons}}) \leq_C w_{\text{total}} + (0^{\text{prod}}, \mathcal{S}(p_{\text{eq}})^{\text{cons}}),$$

i.e., if and only if $\mathcal{Z}(p_{\text{eq}}, q) \leq 0$ *for all* $q \in C^*$*.*

In order to prove existence of such equilibrium pricing functions, and thus prove the main theorem, we construct an auxiliary function $\mathcal{F}$ from the salient half-space C^* to C^*, satisfying

- $\forall p \in C^* \setminus \{0\} : (\exists \alpha \geq 0 : \mathcal{F}(p) = \alpha p) \iff (\forall q \in C^* : \mathcal{Z}(p,q) \leq 0)$.
- $\mathcal{F}$ is continuous on $C^* \setminus \{0\}$.

Obviously, once such a function is obtained, Proposition 1.38 yields the desired result.

In Section 1, it is shown that, under the assumptions of the main theorem, the section $L(x_0) := \{q \in C^* \mid \mathcal{V}(x_0, q) = 1\}$ is compact for every $x_0 \in \text{int}(C)$. For the rather standard way of defining the Lebesgue measure μ on such a section, we also refer to [10].

The finite-dimensionality of C (Assumption A) implies that $\text{int}(C) \neq \emptyset$. Given some fixed $x_0 \in \text{int}(C)$, the function $\mathcal{F}_0 : \text{Domain}^\circ \to C^*$ is defined by

$$\mathcal{F}_0(p) := \int_{L(x_0)} \max\{0, \mathcal{Z}(p,q)\} q \, d\mu(q). \tag{4}$$

Note that

$$\mathcal{Z}(p, \mathcal{F}_0(p)) \geq 0. \tag{5}$$

We extend $\mathcal{F}_0$ to the whole of C^* as follows. The function $\mathcal{F} : C^* \to C^*$ is defined by

$$\mathcal{F}(p) := \begin{cases} (1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{F}_0(p) + \eta(\mathcal{Z}(p, p_0))p_0 & p \in \text{Domain}^\circ \\ p_0 & p \in C^* \setminus \text{Domain}^\circ, \end{cases} \tag{6}$$

where p_0 is any fixed element of int(Domain$^\circ$) (cf. Assumption D) and where η is the sigma-oidal function defined by

$$\eta(\alpha) := \begin{cases} 0 & \text{if } \alpha \leq 0 \\ \alpha & \text{if } 0 < \alpha < 1 \\ 1 & \text{if } 1 \leq \alpha. \end{cases} \tag{7}$$

Note that

$$\forall \alpha \in \mathbb{R} : \alpha\eta(\alpha) \geq 0, \text{ and} \tag{8}$$

$$\alpha\eta(\alpha) = 0 \text{ if and only if } \alpha \leq 0 \tag{9}$$

Lemma 5.4 *Let $p \in C^*$. Then $(\exists\, \alpha \geq 0 : \mathcal{F}(p) = \alpha p) \iff p$ is an equilibrium pricing function.*

Proof
Let $p \in$ Domain$^\circ$ be an equilibrium pricing function. By Lemma 5.3, we find $\forall q \in C^*$: $\mathcal{Z}(p, q) \leq 0$. Hence, by (4), $\mathcal{F}_0(p) = 0$, and by (7), $\eta(\mathcal{Z}(p, p_0)) = 0$. By (6), we conclude that $\mathcal{F}(p) = 0$.
For the converse, suppose $\mathcal{F}(p) = \alpha p$ for some $\alpha \geq 0$. From (6) and the fact that Domain$^\circ \cup \{0\}$ is a cone, it follows that $p \in$ Domain$^\circ$. Walras' law (equation (3)) yields

$$\mathcal{Z}(p, \mathcal{F}(p)) = \alpha\mathcal{Z}(p, p) = 0.$$

By (6), (5) and (8) , we find

$$0 = \mathcal{Z}(p, \mathcal{F}(p)) = \underbrace{(1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{Z}(p, \mathcal{F}_0(p))}_{\geq 0} + \underbrace{\eta(\mathcal{Z}(p, p_0))\mathcal{Z}(p, p_0)}_{\geq 0}.$$

Clearly,

$$(1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{Z}(p, \mathcal{F}_0(p)) = 0 \tag{10}$$

and

$$\eta(\mathcal{Z}(p, p_0))\mathcal{Z}(p, p_0) = 0. \tag{11}$$

By (11) and (9) we find $\mathcal{Z}(p, p_0) \leq 0$, hence, using the definition of η, (10) implies

$$0 = \mathcal{Z}(p, \mathcal{F}_0(p)) = \int_{L(x_0)} \max\{0, \mathcal{Z}(p, q)\}\mathcal{Z}(p, q)d\mu(q).$$

So, for all $q \in L(x_0) : \mathcal{Z}(p, q) \leq 0$. We conclude that p is an equilibrium price functional.
□

In order to prove that the auxiliary function $\mathcal{F}$ is continuous, we need the following lemma.

Lemma 5.5 *The function $\mathcal{F}_0$ is continuous on* Domain$^\circ$.

Proof
Recall the definition of x_0 and $L(x_0)$ in the definition of the auxiliary function $\mathcal{F}$. Impose on C^* the norm $\| \,.\, \|_{x_0}$, and let $\| \,.\, \|$ be the norm on C, dual to the norm $\| \,.\, \|_{x_0}$ (cf. [10]). Thus, by definition, for all $q \in L(x_0)$ we have $\| q \|_{x_0} = 1$. And so, for all $p_1, p_2 \in$ Domain$^\circ$ and $q \in C^*$ we find

$$\begin{aligned}
&| \mathcal{Z}(p_1, q) - \mathcal{Z}(p_2, q) | \\
&= |[\mathcal{D}(p_1), q^{\text{cons}}]_{\text{cons}} - \mathcal{G}(\mathcal{S}(p_1), q) - [\mathcal{D}(p_2), q^{\text{cons}}]_{\text{cons}} + \mathcal{G}(\mathcal{S}(p_2), q)| \\
&\leq \| \mathcal{D}(p_1) - \mathcal{D}(p_2) \| \ + \ \| \mathcal{S}^{\text{cons}}(p_1) - \mathcal{S}^{\text{cons}}(p_2) \| \ + \ \| \mathcal{S}^{\text{prod}}(p_1) - \mathcal{S}^{\text{prod}}(p_2) \| \ .
\end{aligned}$$

From this, and the fact that for all $\alpha, \beta \in I\!R : |\max\{0, \alpha\} - \max\{0, \beta\}| \leq |\alpha - \beta|$, we find

$$\begin{aligned}
&\| \mathcal{F}_0(p_1) - \mathcal{F}_0(p_2) \| \ \leq \int\limits_{L(x_0)} | \max\{0, \mathcal{Z}(p_1, q)\} \ - \ \max\{0, \mathcal{Z}(p_2, q)\} \,| \, d\mu(q) \ \leq \\
&(\| \mathcal{D}(p_1) - \mathcal{D}(p_2) \| \ + \ \| \mathcal{S}^{\text{cons}}(p_1) - \mathcal{S}^{\text{cons}}(p_2) \| \ + \ \| \mathcal{S}^{\text{prod}}(p_1) - \mathcal{S}^{\text{prod}}(p_2) \| \ ) \, \mu(L(x_0)).
\end{aligned}$$

Since $\mathcal{D}$ and $\mathcal{S}$ are continuous on Domain$^\circ$, it follows that $\mathcal{F}_0$ is continuous on Domain$^\circ$.
$\square$

Proposition 5.6 *The function $\mathcal{F} : C^* \setminus \{0\} \to C^*$ is continuous.*

Proof
The function $q \mapsto \eta(\mathcal{Z}(q, p_0))$ is continuous on Domain$^\circ$, and $\mathcal{F}_0$ is continuous on Domain$^\circ$, so the function $\mathcal{F}$ is continuous on Domain$^\circ$. Remains to prove the continuity of $\mathcal{F}$ on $C^* \setminus (\text{Domain}^\circ \cup \{0\})$. By definition, $\mathcal{F}(p) = p_0$ for all $p \in C^* \setminus \text{Domain}^\circ$, so we only have to consider a sequence $(p_n)_{n \in I\!N}$ in Domain$^\circ$ with limit $p \notin \text{Domain}^\circ \cup \{0\}$. Now, suppose the sequence $(\mathcal{F}(p_n))_{n \in I\!N}$ does not converge to p_0. Taking a subsequence if necessary, we may assume $\mathcal{F}(p_n) \neq p_0$, for all $n \in I\!N$. Note that $p \notin$ Domain$^\circ$ means either $p^{\text{cons}} \in \text{bd}(C^*_{\text{cons}})$ or $p^{\text{cons}} \in \text{int}(C^*_{\text{cons}})$ and $\exists j_0 \in \{1, \ldots, J\} : p \notin \text{Domain}[j_0]$.
In the first situation, Corollary 4.15 and Lemma 1.34.b imply $\liminf\limits_{n\to\infty}([\mathcal{D}(p_n), p_0^{\text{cons}}]_{\text{cons}}) = \infty$.
In the second situation, Corollary 3.25 implies $\limsup\limits_{n\to\infty} \mathcal{G}(\mathcal{S}(p_n), p_0) = -\infty$.
Either way, we conclude

$$\liminf_{n\to\infty} \mathcal{Z}(p_n, p_0) = \liminf_{n\to\infty}([\mathcal{D}(p_n), p_0^{\text{cons}}]_{\text{cons}} - \mathcal{G}(\mathcal{S}(p_n), p_0) - \mathcal{V}(w_{\text{total}}, p_0)) = \infty.$$

Hence, $\exists n_0 \in I\!N : \mathcal{Z}(p_{n_0}, p_0) \geq 1$. So, by (6) and (7), $\mathcal{F}(p_{n_0}) = p_0$. This is in contradiction with the assumption that $\mathcal{F}(p_n) \neq p_0$ for all $n \in I\!N$. $\square$

Herewith, applying Proposition 1.38 and Lemma 5.4, we have proved the Existence Theorem.

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